Efficiency-conditional priority to the worst-off

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Abstract

We study how to rank multidimensional allocations, over which agents have heterogeneous preferences, according to a new efficiency-preserving transfer principle. The motivation is that in conjunction with efficiency and informational parsimony principles, transfer requirements previously proposed, which ignore the respective efficiency of the pre-and-post-transfer allocations, force society to give absolute priority to the worst-off. The social orderings that we characterize —on the basis of our new transfer axiom, *strong Pareto*, *continuity* and an assumption pertaining to the representation of individual preferences— combine an inequality-neutral ordering and an infinitely inequality-averse ordering, and may display any finite aversion to inequality, except neutrality.

Keywords: Efficiency; Pareto-Efficiency; Equity; Transfer principle; Social orderings; Opportunity sets.

JEL classification: D61; D63; D71; I31.

1 Introduction

We address the problem of measuring individual and social welfare when evaluating the allocation of resources in a given economy. The purpose of identifying sound social welfare criteria is to inform the public debate and the design of public policies taking into account

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the essential fact that individuals differ in the way they trade off income, leisure, health, education, *etc.*

This objective aligns with several "Beyond GDP" initiatives which heavily insist on the importance of relying on individual values and life plans when weighting different dimensions of living conditions for the sake of evaluating the state of society and various public policies, including redistributive mechanisms. This is the case of the Stiglitz-Sen-Fitoussi Commission (Stiglitz et al. (2009)), the "How is life?" annual report (OECD (2011–2024)), as well as the Pact for the Future that the UN recently adopted (United Nations (2024)). All these initiatives, to various degrees, draw on the literature on well-being, social welfare, and fairness. The need to rely on individual preferences, unlike objective standard measures such as GDP and "mashup indices" (Ravallion (2012)) such as the Human Development Index, was emphasized by both economists and philosophers (Dworkin (1981), Kirman (1992), Fleurbaey (2008), Yang (2018)).

The preference-based orderings of allocations studied here. Our evaluation criteria take the form of Social Ordering Functions (SOFs), which provide an ordering of allocations, as a function of the two elements characterizing an *economy*: the social endowment in commodities and the potentially heterogeneous preferences of agents. This formalism offers a middle ground between purely resourcist approaches and approaches appealing to abstract numbers, "utilities", assumed to express in a unspecified way the satisfaction of individuals. On the one hand, SOFs compare the bundles allocated to individuals on the basis of their preferences.[1] On the other hand, because they rank allocations and not states of purely subjective well-being, they rest on the idea, on which Rawls (1971) and Dworkin (2002) particularly insisted, that social justice is a matter of allocating resources rather than subjective satisfaction or happiness.

A major limitation, for the possibility of policy applications, of part of the literature on fair SOFs —in particular, the central approach developed by Fleurbaey and Maniquet (2011)— is that it *characterizes* SOFs which give *absolute priority to the worst-off*. An example of the induced rankings is a maximin ordering defined on money-metric indices. Condoning arbitrarily large losses for the better-off agents for the sake of tiny gains to the worse-off agents is highly controversial, and one cannot reasonably expect such an extreme form of egalitarianism to be sufficiently widely accepted to guide social evaluations.[2]

In contrast, we characterize SOFs that focus on the worst-off agents *only for equally efficient allocations*, and are compatible with any finite inequality aversion, except neutrality.

[1] Typically, the bundles allocated to two individuals are compared according to a preference-based (ordinal) welfare measure determining which of the two individuals is better-off at their respective bundles.

[2] Even Rawls never argued for the maximin expressed in an individualistic form. He only advocated giving absolute priority to the worst-off social group.

These SOFs combine an inequality-neutral Paretian SOF and an infinitely inequality-averse Paretian SOF.[3] Then, in each economy, allocations are compared according to a monotonic transformation of i) a measure of their efficiency and ii) the minimum welfare level induced by the bundles they assign, according to a specific mapping described later.

Formally, for a given economy, in which a group N of agents have preferences $R_N = (R_i)_{i \in N}$, allocations are compared according to a mapping λ^{R_N} measuring their efficiency and an individual welfare index mapping w^{R_i}, for all $i \in N$. An example of the latter is a money-metric index mapping, computed at a reference price which is fixed independently of the preferences of individuals.[4] Then, there is a continuous mapping F, increasing in each argument, such that an allocation $x = (x_i)_{i \in N}$ is socially considered at least as desirable as an allocation $y = (y_i)_{i \in N}$ if and only if $F\Big(\lambda^{R_N}(x), \min_{i \in N} w^{R_i}(x_i)\Big) \geq F\Big(\lambda^{R_N}(y), \min_{i \in N} w^{R_i}(y_i)\Big)$. Inequality aversion —the extent to which society prioritizes equity over efficiency— is expressed here in a clear-cut manner.

An important consequence is that any two equally-efficient allocations are compared exclusively on the basis of the minimal welfare level attained. According to a ranking as above, starting from an allocation x of the social endowment, an *efficiency-neutral* re-allocation y of the endowment is preferred to x only if it benefits the worst-off agents, but a *Pareto-improving* re-allocation z of the endowment is always preferred to x. We describe this property as an *efficiency-conditional priority to the worst-off.*

Crucially, since two Pareto-efficient allocations of an economy are considered equally efficient by the inequality-neutral SOF —we elaborate on its properties below— they are compared focusing exclusively on the situation of the worst-off agents. This guarantees that on the Pareto-frontier of an economy, our rankings satisfy an important equity principle: a multi-commodity transfer *à la Pigou–Dalton* between two agents with the same preference relation is a (weak) social improvement. In contrast, while the motivation of this paper is close to that of Bosmans et al. (2018), the social welfare criteria that they characterize violate this equity property. In this light, we now turn to the normative requirements around which our analysis is structured.

The axiomatic analysis. In line with the previous observations on the importance of respecting individual preferences in the evaluation of allocations, we work under the

[3]By “Paretian”, we refer here to the satisfaction of weak Pareto —our axiomatic approach is described below.

[4]Then, the number $w^{R_i}(x_i)$ is the minimum income, computed at this reference price, that agent i needs in order to reach a bundle that she finds at least as desirable as the bundle x_i.

standard strong Pareto axiom.[5] In a multidimensional framework, this principle proves *hard* to combine with an equity requirement expressing a preference for balanced, rank-preserving and progressive transfers (Pigou (1912), Dalton (1920)). Consider a redistribution of positive quantities of each good from one agent to an agent who has less of each good, such that, after the transfer, the initially richer agent still consumes more of every good than the recipient. These agents have the same preference relation, so that the richer agent is unambiguously better-off. The other agents are unaffected. Then, same-preference transfer requires that the final allocation be socially at least as desirable as the initial allocation.[6]

Yet, as we alluded to above, *same-preference transfer* forces the social ranking to give absolute priority to the worst-off, when assumed together with *strong Pareto* and the following informational parsimony requirement (Fleurbaey and Maniquet (2011)). It is a relaxation of Arrow's independence axiom and states that the ranking of two allocations should depend, "at most", on each agent's indifference curves *through her bundles at these allocations* —this is unchanged-contour independence below— and on the resources that are available in the economy.[7]

Precisely, a core motivation of this work is to avoid such an extreme attitude toward inequality, building on a (more) compelling fairness principle, without giving ground on efficiency and informational parsimony. This view calls for weakening *same-preference transfer*.

We follow this path, inspired by the case made by Bosmans et al. (2018) in favor of restricting attention to multidimensional "efficiency-neutral transfers", given that, in the single-good case that we seek to generalize, a Pigou–Dalton transfer necessarily preserves efficiency. The underlying idea is that an allocation which is at least "as efficient" as another allocation, and more equitable, in the sense that it is obtained from this second allocation through a same-preference transfer, is unequivocally socially at least as desirable. Accordingly, the way in which efficiency is measured is fundamental.

In our view, if we are to condition the desirability of transfers on the efficiency of the pre-and-post-transfer allocations, then the case of two Pareto-efficient allocations related

[5] It requires an allocation to be ranked at least as high as another one whenever all agents consider their bundle in the first allocation to be at least as desirable as their bundle in the other one. It also requires the first allocation to be socially preferred as soon as, additionally, at least one agent prefers her bundle in the first allocation. It expresses both an anti-paternalism concern, and anti-waste one: the view is that the allocation of resources should allow individuals to follow their own conception of the good life, and that opportunities for individuals to achieve their goals should not be wasted.

[6] We discuss the same-preference assumption in detail in Section 3. The axiom "defined" by dropping this assumption in the description above is incompatible with *strong Pareto*.

[7] *Unchanged-contour independence* is consistent with the way representations of individual preferences are used in the measurement of inequality and social welfare, as well as in *cost-benefit analysis* (Deaton (1979), Deaton and Muellbauer (1980), King (1983), Layard et al. (1994), Fleurbaey et al. (2013), Decancq and Schokkaert (2016), Jones and Klenow (2016), Hammitt (2021), De Rock and Moramarco (2025), Brouillette et al. (2025)).

by a transfer is the case in point. In line with the theory of fair allocation, which seeks to *select* allocations of resources that are Pareto-efficient, and, in some sense, fair (see Thomson (2011) for a review), it is important, when studying *ranking* criteria, to respect a transfer principle along the Pareto-frontier of economies.

As a consequence, our same-preference efficiency-neutral transfer axiom is defined by adding to the assumptions of *same-preference transfer* the restriction that the pre-and-post transfer allocations are equally efficient according to a SOF with the following properties. Two allocations with the same *Scitovsky boundary*, which is the case to which the *weaker* transfer axiom in Bosmans et al. (2018) is restricted, are considered equally efficient. Furthermore, two Pareto-efficient allocations, potentially with different Scitovsky boundaries, are also considered equally efficient.

We work under an additional assumption that weakens the condition imposed on interpersonal comparisons in Bosmans et al. (2018), which clashes with *strong Pareto* and our transfer axiom.[8] Namely, we assume that for equally efficient allocations, interpersonal comparisons are pursued only on the basis of a reference-set welfare index.[9]

At first, we assume that the reference sets are *exogenous*, *i.e.* fixed independently of the preferences of agents and the social endowment. Our first characterization of SOFs admitting the representation described above involves this assumption, *strong Pareto*, continuity and *same-preference efficiency-neutral transfer* (Theorem 1). Then, we extend the analysis by allowing the choice of the reference sets to depend on the social endowment (Theorem 3). This allows, for instance, the use of money-metric indices computed for a reference price reflecting the relative available quantities. The last of our positive results is a characterization of SOFs that *rationalize* the *egalitarian-equivalent correspondence*, one of the two core solutions studied in the theory of fair allocation (Theorem 4).

We introduce the model in Section 2, and the axioms in Section 3. Results are presented and discussed in Sections 4 and 5. Section 6 concludes. Most of the proofs are gathered in the Appendix.

[8] More precisely, combined with *strong Pareto* and our transfer axiom, it implies an infinite inequality aversion, which is a consequence of our Theorem 1.

[9] Examples of such indices include money-metric indices and ray indices, which are widely used. In a nutshell, a family of nested (reference) sets of bundles is indexed by the non-negative reals. Then, the welfare level of an agent at a given bundle is identified to the index number associated to the set which is "tangent" to her indifference curve at this bundle.

2 Preliminaries

2.1 General notation

The set of real numbers, the set of non-negative real numbers, and the set of positive real numbers are $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$, respectively. The set of natural numbers is $\mathbb{N}$. Component-wise vector inequalities are denoted by $\geqq, \geq, >$.[10] The standard inner product operator is denoted by the symbol $\cdot$. A vector of equal coordinates is denoted by a boldface character. The closed line segment connecting two vectors of same dimension x and y is denoted by $\text{seg}[x, y]$. The ray with origin at x going through $y \neq x$ is denoted by $\text{ray}[x, y)$.

The cardinality of a set A is denoted by $|A|$. Weak set inclusion is denoted by $\subseteq$ and strict set inclusion by $\subsetneqq$. The Cartesian product of A and B is denoted by $A \times B$, and, for any $k \in \mathbb{Z}_{++}$, A^k denotes the k-fold Cartesian product of A. The (Minkowski) addition of sets is defined as $A + B = \{x \mid \exists (a, b) \in A \times B, x = a + b\}$. The set A^B is the set of mappings from B to A. The open ball of radius $\epsilon > 0$ around $x \in \mathbb{R}^k$ is denoted by $B(x, \epsilon)$. For two subsets A and B of a topological set, the boundary of A relative to B is denoted by $\partial_B A$, and, when the set relative to which the topological boundary is clearly identified, we simply write ∂A.[11]

We use the symbol $\neg$ to denote logical negation.

A **quasi-ordering** on a set A is a reflexive and transitive binary relation on A, and an **ordering** on A is a complete quasi-ordering on A. The subset of maximal elements of a set $B \subseteq A$ for a quasi-ordering R is denoted by $\max|_R B$.

2.2 Basic elements of the model

There is a set L of goods, with $|L| \geq 2$. The social endowment of goods is denoted by $\Omega = (\Omega^l)_{l \in L} \in \mathbb{R}^L_{++}$. The population is a finite set N, with $|N| \geq 2$.

Each agent i in N has a **preference relation** R_i, which is an ordering over bundles z_i belonging to agent i's consumption set $X = \mathbb{R}^L_+$. For two bundles $x_i = (x_i^l)_{l \in L}, y_i = (y_i^l)_{l \in L}$, we write $x_i R_i y_i$ to indicate that agent i considers x_i at least as desirable as y_i. The corresponding strict preference and indifference relations are denoted by P_i and I_i respectively.[12] We restrict attention to orderings that are **continuous** —for all $x_i \in X$, the upper-contour set $U(x_i, R_i) = \{y_i \in X \mid y_i R_i x_i\}$ and the lower-contour set $L(x_i, R_i) = \{y_i \in X \mid x_i R_i y_i\}$,

[10] For two vectors of same dimension $x = (x_1, \cdots, x_K)$ and $y = (y_1, \cdots, y_K)$, $x \geqq y$ if $x_k \geq y_k$ for all $1 \leq k \leq K$; $x \geq y$ if $x \geq y$ and $x \neq y$; $x > y$ if $x_k > y_k$ for all $1 \leq k \leq K$.

[11] That is, $\partial_B A = \overline{A}_B \cap (\overline{B \setminus A})_B$, where $\overline{A}_B$ is the closure of A relative to B and $(\overline{B \setminus A})_B$ is the closure of $B \setminus A$ relative to B.

[12] For all $x_i, y_i \in X^N$, $x_i P_i y_i$ if and only if $[x_i R_i y_i]$ and $\neg[y_i R_i x_i]$, and $x_i I_i y_i$ if and only if $[x_i R_i y_i]$ and $[y_i R_i x_i]$.

at x_i, are closed—, **monotonic** —for all $x_i, y_i \in X$, if $x_i \geqq y_i$, then $x_i R_i y_i$ and if $x_i > y_i$, then $x_i P_i y_i$—, and **convex** —for all $x_i, y_i \in X$, if $x_i R_i y_i$ then $(\lambda x_i + (1-\lambda) y_i)\ R_i\ y_i$ for all $\lambda \in [0,1]$.[13] Let $\mathcal{R}$ denote the set of such preferences.

For the preference relation R_i, the indifference set of agent i with preference $R_i \in \mathcal{R}$ at bundle $x_i \in X$ is $I(x_i, R_i) = \{y_i \in X \mid y_i I_i x_i\}$. We let $P_{\text{supp}}(x_i, R_i)$ denote the set of supporting prices of $U(x_i, R_i)$ at x_i.[14]

An **economy** is denoted by $E = (R_N, \Omega)$, where $R_N = (R_i)_{i \in N} \in \mathcal{R}^N$ is the profile of preferences for the whole population.[15] Let $\mathcal{E}$ denote the class of all economies satisfying the preceding conditions. An **allocation** is a list of bundles $x_N = (x_i)_{i \in N} \in X^N$. It is **feasible** for E if

$$\sum_{i \in N} x_i \leqq \Omega.$$

We denote the set of feasible allocations for E by $\text{Fea}(E)$.

A **social ordering** (for economy $E = (R_N, \Omega)$) is an ordering over the set X^N. A **Social Ordering Function (SOF) R** associates to every economy E in some domain $\mathcal{D} \subseteq \mathcal{E}$ a social ordering $\mathbf{R}(E)$. A SOF that is introduced without specifying any domain of economies should be assumed to be defined on $\mathcal{E}$. For an economy E and two allocations $x_N, y_N \in X^N$, we write $x_N \mathbf{R}(E) y_N$ to indicate that x_N is socially at least as desirable as y_N for E. We thus study social orderings that are not merely defined over the set of feasible allocations. However, it will be clear that our analysis would not be changed if we adopted this alternative modeling, assuming in addition, *for some* results, a mild requirement pertaining to the sensitivity of social orderings to the social endowment.[16] The strict social preference and social indifference relations associated to $\mathbf{R}(E)$ are denoted by $\mathbf{P}(E)$ and $\mathbf{I}(E)$ respectively.

3 Axioms

We use bold letters to name axioms that we impose, and italic letters to name properties that we simply discuss. Properties may be imposed on any domain $\mathcal{D} \subseteq \mathcal{E}$.

[13] The definitions of a strictly monotonic and strictly convex ordering are obtained by replacing R_i by P_i in the preceding definitions, and taking $\lambda \in]0,1[$ for a strictly convex ordering.

[14] $p \in \mathbb{R}^L_+ \setminus \{\mathbf{0}\}$ is a supporting price of $U(x_i, R_i)$ at x_i if the hyperplane $H = \{z_i \in X | \ p \cdot z_i = p \cdot x_i\}$ supports $U(x_i, R_i)$ at x_i, that is, $p \cdot y_i \geq p \cdot x_i$ for all $y_i \in U(x_i, R_i)$.

[15] For all non-empty, $S \subsetneq N$, and all $R_N, R'_N \in \mathcal{R}^N$, (R_S, R'_{-S}) denotes $R''_N = ((R_i)_{i \in S}, (R'_j)_{j \in N \setminus S})$. The same convention is adopted for allocations.

[16] The property is the following. *Independence from proportional expansions of feasible allocations* states that a social ordering may depend on the social endowment, but only through the relative scarcity of resources: let $E = (R_N, \Omega) \in \mathcal{D}$, $\mu > 1$ and $E^\mu = (R_N, \mu\Omega) \in \mathcal{D}$. Let $x_N, y_N \in \text{Fea}(E)$. Then, $x_N \mathbf{R}(E) y_N \iff x_N \mathbf{R}(E^\mu) y_N$.

3.1 Three standard axioms

We study SOFs satisfying the strongest version of the Pareto principle.

Strong Pareto

Let $E = (R_N, \Omega) \in \mathcal{D}$ and $x_N, y_N \in X^N$ be such that, for all $i \in N$, $x_i R_i y_i$. Then, $x_N \mathbf{R}(E) y_N$. In addition, if there is i such that $x_i P_i y_i$, then $x_N \mathbf{P}(E) y_N$.

In the latter case, x_N *Pareto-dominates* y_N. The set of **Pareto-efficient** allocations of an economy $E = (R_N, \Omega)$, that is, the set of allocations $x_N \in \text{Fea}(E)$ such that there is no $y_N \in \text{Fea}(E)$ that Pareto-dominates x_N, is denoted by $\text{Par}(E)$. Two allocations $x_N, y_N \in X^N$ are Pareto-equivalent if, for all $i \in N$, $x_i I_i y_i$. The *strong Pareto* axiom implies that the social ranking does not differentiate between two Pareto-equivalent allocations, a property to which we refer by *respect of Pareto equivalence*.[17]

We also require that a SOF not be biased towards any agent in the following sense:

Anonymity

Let $E = (R_N, \Omega) \in \mathcal{D}$ be such that $R_i = R_j$ for some $i, j \in N$. Let x_N and $y_N \in X^N$ be such that

$$x_k = y_k \text{ for all } k \in N \setminus \{i, j\},$$
$$x_i = y_j \text{ and } x_j = y_i.$$

Then, $x_N \mathbf{I}(E) y_N$.

For some of our results, *anonymity* follows from the other axioms that we adopt.

We impose *continuity* on the social ordering associated with each economy, so that small changes in allocations do not lead to large changes in their social evaluation:

Continuity

Let $E = (R_N, \Omega) \in \mathcal{D}$, and $x_N \in X^N$. Then, the sets $\{y_N \mid y_N \mathbf{R}(E) x_N\}$ and $\{y_N \mid x_N \mathbf{R}(E) y_N\}$ are closed.

[17] Let us give formal definition of *Weak-Pareto*. Let $E = (R_N, \Omega) \in \mathcal{D}$, and $x_N, y_N \in X^N$ such that, for all $i \in N$, $x_i R_i y_i$. Then, $x_N \mathbf{R}(E) y_N$.

3.2 An efficiency-preserving transfer axiom covering the case of Pareto-efficient allocations

Our next axiom pertains to the aversion to inequality of a SOF. It expresses a preference for balanced, rank-preserving, and progressive transfers, involving agents with the same preference relation, and relating "equally efficient" allocations.

Focusing on balanced, rank-preserving, and progressive transfers follows the general idea of Pigou (1912) and Dalton (1920). We claim in the following that our multidimensional setting calls for the two other restrictions.

3.2.1 Restricting to transfers among agents with disjoint indifference curves

First of all, the immediate generalization of the Pigou-Dalton principle, defined below, is incompatible with *strong Pareto* (Fleurbaey and Trannoy (2003)).

Transfer
Let $E = (R_N, \Omega) \in \mathcal{D}$. Let $x_N, y_N \in X^N$ be such that, for some $i \neq j \in N$,

$$x_k = y_k \text{ for all } k \in N \setminus \{i, j\},$$
$$y_i - \Delta = x_i > x_j = y_j + \Delta, \text{ for some } \Delta \in \mathbb{R}^L_{++}.$$

Then, $x_N \mathbf{R}(E) y_N$.

Consider a redistribution of positive quantities of each good from one agent to an agent who has less of each good, such that, after the transfer, the initially richer agent still consumes more of every good than the recipient. Assume, in addition, that the other agents are unaffected. Then, *transfer* requires that the final allocation be socially at least as desirable as the initial allocation.

In a nutshell —we refer the reader to Fleurbaey and Trannoy (2003)— a cycle (*i.e.* a violation of transitivity) in the social ranking induced by a SOF satisfying *transfer* and *strong Pareto* may arise when the indifference curves of i and j, at x_i and x_j, and at y_i and y_j, cross.[18]

The following weakening of *transfer*, which adds the restriction that the agents involved in the transfer should have the same preference relation, is compatible with *strong Pareto.*

[18]The reason behind is that when these indifference curves cross, while both agents prefer x_i to x_j, there may be a bundle on the indifference curve of j at x_j that dominates a bundle on the indifference curve of i at x_i. That is, while $x_i > x_j$, there may be bundles $z_i, z_j \in X$ such that $x_i I_i z_i$, $x_j I_j z_j$ and $z_i < z_j$.

It has been adopted in a number of important studies.[19] The fact that these agents have the same preference relation guarantees not only that both prefer the dominating bundle, but also that the same preference holds for all Pareto-equivalent allocations. In other words, the indifference curves of the agents at the their bundles in the pre-and-post transfer allocations do not cross. This prevents the type of cycle we alluded to.

Same-preference transfer (Fleurbaey and Maniquet (2011))

Let $E = (R_N, \Omega) \in \mathcal{D}$ be such that $R_i = R_j$ for some $i \neq j \in N$. Let $x_N, y_N \in X^N$ be such that

$$
\begin{aligned}
&x_k = y_k \text{ for all } k \in N \setminus \{i, j\},\\
&y_i - \Delta = x_i > x_j = y_j + \Delta, \text{ for some } \Delta \in \mathbb{R}^L_{++}.
\end{aligned}
$$

Then, $x_N \mathbf{R}(E) y_N$.

Our transfer axiom is also restricted to agents with the same preference relation and it is weaker than *same-preference transfer*. It accommodates the possibility of a trade-off between equity and efficiency in the evaluation of transfers, something which *same-preference transfer* ignores.

3.2.2 Accounting for efficiency when evaluating a transfer

In the one-dimensional case, any two allocations whose components add up to the same amount, such as allocations related by a balanced transfer, are unambiguously equally efficient. In contrast, when multiple resources are to be distributed, while the allocation that results from a transfer among agents with the same preference relation may be fairer than the initial one, it may be (much) less efficient. This observation is what motivates the restriction to transfers along a line segment, among agents with the same *homothetic* preference relation, in Fleurbaey and Tadenuma (2014). Then, one can conclude, with Bosmans et al. (2018), that by imposing no condition on the efficiency of the allocations related by a transfer, *same-preference transfer* expresses an "extreme" stance in favor of equity.

It should be noted that, by itself, *same-preference transfer* is compatible with any inequality aversion, including neutrality. Furthermore, by allowing the social ordering to depend on preference profiles in an arbitrarily fine way, that is, by imposing no independence axiom, Piacquadio (2017) has characterized an important class of orderings satisfying, in particular, *strong Pareto* and *same-preference transfer*.

[19] See Fleurbaey and Maniquet (2011), Fleurbaey and Tadenuma (2014), Piacquadio (2017), Bosmans et al. (2018).

This being said, *same preference transfer* indeed ignores the potential trade-off between efficiency and equity when evaluating a multidimensional transfer, so much so that, combined with *respect of Pareto equivalence* and *unchanged-contour independence*[20] or, *a fortiori*, with the stronger "welfarism axiom" that Bosmans et al. (2018) adopt (see Section 3.3 for a definition), it implies an infinite aversion to inequality (Fleurbaey and Maniquet (2011)). Yet, the conclusion that allocation policies should be guided by a criterion that condones arbitrarily large losses for the better-off agents for the sake of tiny gains to the worse-off agents is highly dubious.

This discussion provides a strong rationale for the adoption of a transfer axiom which enables to disentangle the objective of equity from the objective of efficiency. This is precisely what our axiom does: akin to the unidimensional Pigou-Dalton axiom, it only concerns transfers between "equally efficient" allocations. Accordingly, the way in which we measure efficiency is crucial, and is what differentiates our axiom from that of Bosmans et al. (2018), which shares the same motivation, but is weaker. In particular, the notion of Pareto-efficiency, one of the cornerstones of economic analysis, is so fundamental that, in our view, one must use a measure of efficiency that treats any two Pareto-efficient allocations as equally efficient. In turn, our axiom covers the case of any two Pareto-efficient allocations related by a same-preference transfer, in contrast to their axiom.

Before stating our transfer axiom, we need to introduce the measure of efficiency on the basis of which it is defined. The **resource utilization SOF**, due to Debreu (1951), denoted by $\mathbf{R}_{\text{eff}}$, maps any economy $E = (R_N, \Omega) \in \mathcal{D}$ to the following ordering:

$$\begin{gathered} x_N \mathbf{R}_{\text{eff}}(E) y_N \\ \iff \lambda^E(x_N) \geq \lambda^E(y_N), \end{gathered}$$

where, for all $z_N \in X^N$, $\lambda^E(z_N) \in \mathbb{R}_+$ is the non-negative real number such that there exists $z'_N \in \text{Par}((R_N, \lambda^E(z_N)\Omega))$ such that, for all $i \in N$, $z_i I_i z'_i$.

In words, the scalar $\lambda^E(z_N)$ is the fraction λ of the social endowment Ω which would suffice to provide the same satisfaction level as in z_N to all agents.[21]

Figure 1 illustrates the definition of $\lambda^E : \mathbb{R}_+^{2\times 2} \to \mathbb{R}_+$, for an economy E composed of

[20]Here is a formal definition of *unchanged-contour independence*. Let $E = (R_N, \Omega)$ and $E' = (R'_N, \Omega)$ in $\mathcal{D}$. Let x_N, y_N in X^N be such that, for all $i \in N$,

$$I(x_i, R_i) = I(x_i, R'_i) \text{ and } I(y_i, R_i) = I(y_i, R'_i).$$

Then, $x_N \mathbf{R}(E) y_N \iff x_N \mathbf{R}(E') y_N$.

[21]That is, in an economy with a social endowment of $\lambda\Omega$, there exists a Pareto-efficient allocation that is Pareto-equivalent to z_N. An equivalent definition of $\lambda^E(z_N)$, as a sum of special money-metric indices, is given in Section 3.3.

two agents, i and j, with preferences $R_i, R_j \in \mathcal{R}$, and two goods, 1 and 2. Good 1 and 2 are available in quantities $\Omega^1, \Omega^2 > 0$, and we consider an allocation $z_N \in X^N$ that is not Pareto-efficient in E. The allocation $z'_N \in X^N$ is Pareto-efficient in the economy E' in which preferences are the same as in E and the social endowment is $\frac{3}{4}\Omega$. In addition, z'_N is Pareto-equivalent to z_N in E (and E'). Thus, $\lambda^E(z_N) = \frac{3}{4}$.

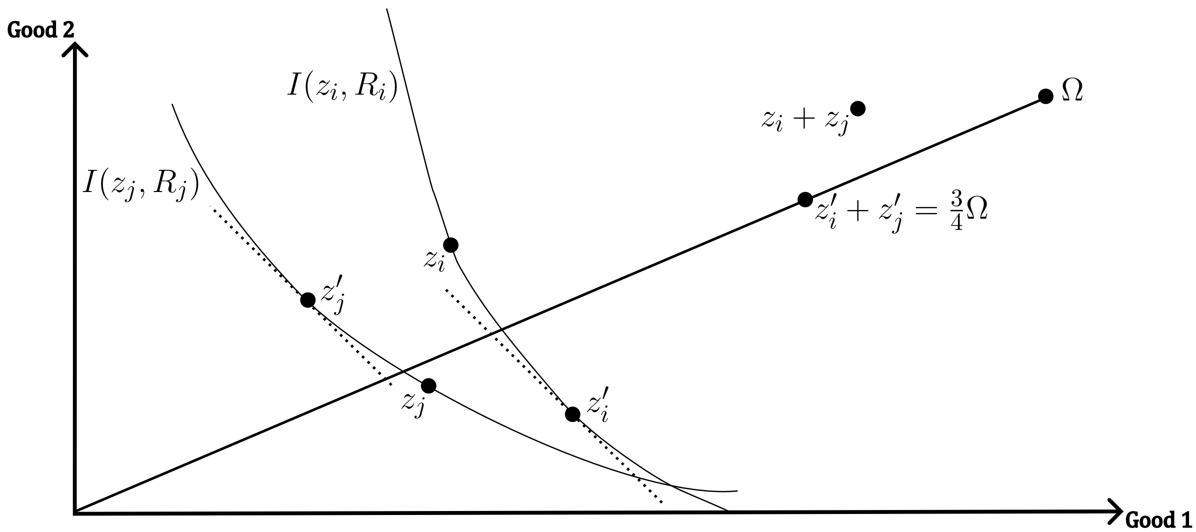


Figure 1. The resource utilization SOF.

Note that $\mathbf{R}_{\text{eff}}$ satisfies *strong Pareto*, which is essential for the desirability of the SOFs we characterize in Section 4.

We can now state our transfer axiom.

Same-preference efficiency-neutral transfer (with respect to $\mathbf{R}_{\text{eff}}$)

Let $E = (R_N, \Omega) \in \mathcal{D}$ be such that $R_i = R_j$ for some $i \neq j \in N$. Let $x_N, y_N \in X^N$ be such that $x_N \mathbf{I}_{\text{eff}}(E) y_N$ and

$$x_k = y_k \text{ for all } k \in N \setminus \{i, j\},$$
$$y_i - \Delta = x_i > x_j = y_j + \Delta, \text{ for some } \Delta \in \mathbb{R}^L_{++}.$$

Then, $x_N \mathbf{R}(E) y_N$.

Same-preference efficiency-neutral transfer implies, in contrast to what can be achieved in the framework of Bosmans et al. (2018), that any same-preference transfer relating Pareto-efficient allocations either improves or leaves unchanged social welfare. In our view, if we are to condition the desirability of transfers on the efficiency of the pre-and-post-transfer allocations, then the case of two Pareto-efficient allocations is the case in point. In line with the theory of fair allocation, which seeks to *select* allocations of resources that are Pareto-efficient, and, in some sense, fair (see Thomson (2011) for a review), it is important, when studying *ranking* criteria, to respect a transfer principle along the Pareto-frontier of economies.

Beyond allocations with the same Scitovsky set. *Same-preference efficiency-neutral transfer* is stronger than the transfer axiom in Bosmans et al. (2018), which is restricted to allocations with the same Scitovsky set.

The **Scitovsky set** of an allocation $x_N \in X^N$ in E is the Minkowski sum of the individual upper-contour sets, that is, the set $S^E(x_N) = U(x_1, R_1) + \cdots + U(x_{|N|}, R_{|N|})$. Then, $\omega \in \mathbb{R}^L_+$ belongs to the (topological) **Scitovsky boundary** of x_N if and only if there exists y_N such that $\sum_i y_i \leqq \omega$ and $y_i I_i x_i$ for all $i \in N$, and there does not exist $w' \in \mathbb{R}^L_+$ and $y'_N \in \mathbb{R}^L_+$ such that $w' \leq w$, $\sum_i y'_i \leqq \omega$ and $y'_i I_i x_i$ for all $i \in N$. In words, ω is minimal, for the usual partial order of $\mathbb{R}^L_+$, among vectors of available resources that can be allocated in a way that leaves every agent indifferent or better-off as compared to her bundle in x_N.[22] As a consequence, $x_N \in \mathrm{Par}(E)$ if and only if $\Omega \in \partial S^E(x_N)$.

Crucially, two allocations with the same Scitovsky boundary in E are considered equally efficient according to $\mathbf{R}_{\text{eff}}(E)$, *but it is also the case of two Pareto-efficient allocations in E with different Scitovsky boundaries.*

Some dependence on the available quantities. In general, for $R_N \in \mathcal{R}^N$, $\Omega_1, \Omega_2 \in \mathbb{R}^L_{++}$, $\mathbf{R}_{\text{eff}}((R_N, \Omega_1))$ is different from $\mathbf{R}_{\text{eff}}((R_N, \Omega_2))$; our measure of efficiency depends on the social endowment in the economy. We contrasted our axiom with "what can be achieved" in the framework of Bosmans et al. (2018): their model can be described as requiring that a SOF depend only on the preference profile under consideration, and not on the social endowment Ω.[23] Under this independence requirement, the fact that the set of Pareto-efficient allocations of an economy depend on the social endowment precludes the adoption of an axiom covering the case of any two Pareto-efficient allocations related by a same-preference transfer.

Under the mild requirement of *independence from proportional expansions*, according to

[22] Let us call their axiom *same-preference transfer for allocations with same Scitovsky set.* Let $E = (R_N, \Omega) \in \mathcal{D}$ be such that $R_i = R_j = R_0$ for some $i, j \in N$. Let $x_N, y_N \in X^N$ be such that $U_i(x_i, R_0) + U_j(x_j, R_0) = U_i(y_i, R_0) + U_j(y_j, R_0)$, and

$$x_k = y_k \text{ for all } k \in N \setminus \{i, j\},$$
$$y_i - \Delta = x_i > x_j = y_j + \Delta, \text{ for some } \Delta \in \mathbb{R}^L_{++}.$$

Then, $x_N \mathbf{R}(E) y_N$.

[23] Let us call this requirement *independence from available quantities.* Let $E = (R_N, \Omega), E' = (R_N, \Omega') \in \mathcal{D}$. Then, $x_N \mathbf{R}(E) y_N \iff x_N \mathbf{R}(E') y_N$.

which a SOF should depend on the social endowment only through the relative available quantities, $\mathbf{R}_{\text{eff}}$, almost by definition, stands out, among plausible measures of efficiency satisfying *strong Pareto*, as the *finest* one.

Independence from proportional expansions

Let $E = (R_N, \Omega) \in \mathcal{D}$, $\mu > 1$ and $E^\mu = (R_N, \mu\Omega) \in \mathcal{D}$. Let $x_N, y_N \in X^N$. Then, $x_N \mathbf{R}(E) y_N \iff x_N \mathbf{R}(E^\mu) y_N$.

For a SOF $\tilde{\mathbf{R}}_{\text{eff}}$ to qualify as a measure of efficiency, a minimal requirement is that, in any economy, the resulting measure be maximized, over the set of feasible allocations, by all Pareto-efficient allocations: for all $E = (R_N, \Omega) \in \mathcal{D}$, $\text{Par}(E) \subseteq \max|_{\tilde{\mathbf{R}}_{\text{eff}}(E)} \text{Fea}(E)$. In the following statement, a **plausible** measure of efficiency is a SOF satisfying this maximization property. Note that if $\tilde{\mathbf{R}}_{\text{eff}}$ additionally satisfies *respect of Pareto equivalence*, then $\text{Par}(E) \subsetneq \max|_{\tilde{\mathbf{R}}_{\text{eff}}(E)} \text{Fea}(E)$.

Proposition 1. *Let $\tilde{\mathbf{R}}_{\text{eff}}$ be a plausible measure of efficiency defined on $\mathcal{E}$ satisfying respect of Pareto equivalence and independence from proportional expansions. Let $E = (R_N, \Omega)$, and let $x_N, y_N, x'_N, y'_N \in X^N$ be such that $x_N \mathbf{I}_{\text{eff}}(E) y_N$ and $x'_N \mathbf{P}_{\text{eff}}(E) y'_N$. Then, $x_N \tilde{\mathbf{I}}_{\text{eff}}(E) y_N$ and $x'_N \tilde{\mathbf{R}}_{\text{eff}}(E) y'_N$.*

Proof. Let $\lambda = \lambda^E(x_N) = \lambda^E(y_N)$. Let $E^\lambda = (R_N, \lambda\Omega)$. By definition of λ^E, and by *respect of Pareto equivalence*, x_N and y_N both belong to $\max|_{\tilde{\mathbf{R}}_{\text{eff}}(E^\lambda)} \text{Fea}(E^\lambda)$. *Independence from proportional expansions* then implies $x_N \tilde{\mathbf{I}}_{\text{eff}}(E) y_N$. The other assertion is proved in the same way. □

3.3 A common individual welfare index to compare equally efficient allocations

The vast majority of approaches that construct social welfare rankings on the basis of ordinal and non-comparable preferences use *reference sets* in order to build individual welfare indices. It is then tempting to require that, for all possible economies, the same reference sets and the same ordering on the associated indices be used to compare allocations. Yet, one consequence of our axiomatic analysis (Lemma 2 in the Appendix) is that such a requirement is incompatible with *strong Pareto*, *same-preference efficiency-neutral transfer*, and a *finite* inequality aversion.

3.3.1 Definition of reference sets

A family of **reference sets** is a family $S = (S_\gamma)_{\gamma \in \mathbb{R}_+}$ of strictly nested compact subsets of X (that is, $\gamma < \mu$ implies $S_\gamma \subsetneqq S_\mu$ and $\partial S_\gamma \cap \partial S_\mu = \emptyset$) such that $S_0 = \{0\}$ and $\bigcup_{\gamma \in \mathbb{R}_+} \partial S_\gamma = X$. Additionally all sets are lower-comprehensive (if x_i belongs to S_γ, then all bundle $y_i \leqq x_i$ belong to S_γ). In the following, we keep the notation that we have been using throughout the paper to denote bundles, *i.e.* x_i, rather than using x, even if reference sets are common to all agents. Let $\mathcal{S}$ be the set of all such lists.

For $S \in \mathcal{S}$, the index assigned to a pair (x_i, R_i) is the greatest number γ for which $x_i R_i y_i$ for all $y_i \in S_\gamma$. Accordingly, the reference-set index w_S is defined by

$$w_S(x_i, R_i) = \max\{\gamma \mid x_i R_i S_\gamma\} \text{ for all } x_i \in X \text{ and all } R_i \in \mathcal{R}.$$

Figure 2 illustrates this definition in the case $|L| = 2$.

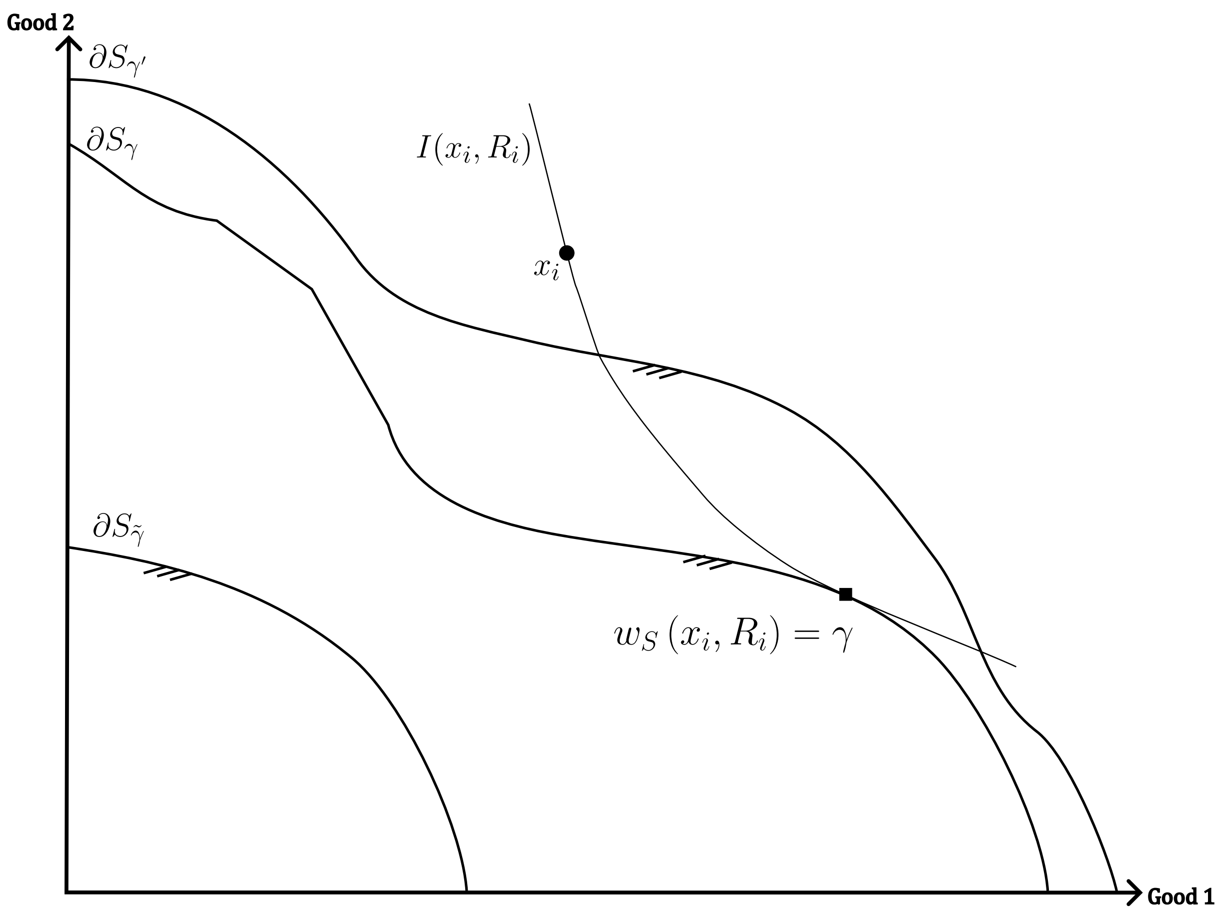


Figure 2. Reference-set index.

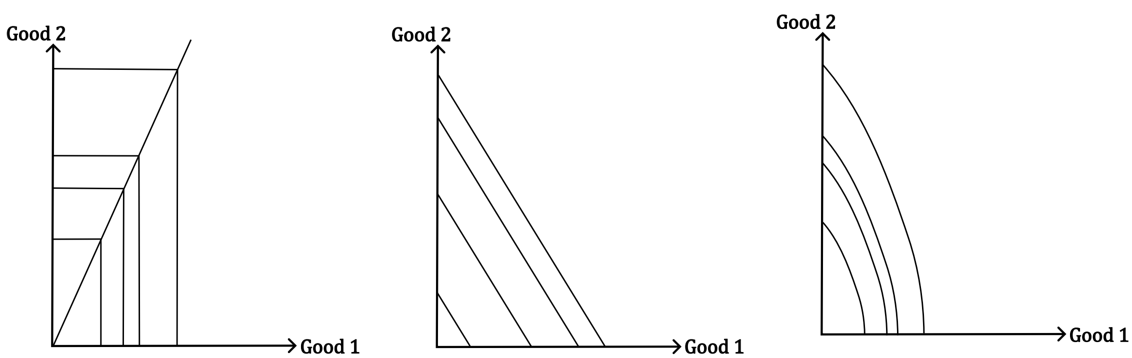
The properties of the preferences in $\mathcal{R}$ and of the lists of reference sets in $\mathcal{S}$ ensure that the reference-set index mapping w_S is well defined, unique, and that its projection on

X is continuous. One obtains a representation of R_i, *i.e.*, for all bundles x_i and y_i in X, we have $w_S(x_i, R_i) \geq w_S(y_i, R_i)$ if and only if $x_i R_i y_i$. We sometime simplify the notation $(w_S(x_i, R_i))_{i \in N}$ to $w_S^{R_N}(x_N)$. In addition, for all $z_i \in X$, we let $S(z_i)$ be the reference set such that $z_i \in \partial S(z_i)$ —note that it is uniquely defined.

Reference-set welfare indices are treated as interpersonally comparable: in a given allocation, two individuals are considered equally well-off if there is a common choice set such that each of them is as well-off at their component of the allocation as at her favorite bundle in this choice set (Thomson (1994)).

The two prominent types of reference sets define **ray-indices** and **money-metric indices**, respectively. Let $\omega \in \mathbb{R}^L_{++}$, and define, for all $\gamma \in \mathbb{R}_+$, $S_\gamma = \{x_i \in X \mid x_i \leqq \gamma\omega\}$. The resulting reference-set mapping is called the **ω-ray index mapping** and we denote it by w_ω. Let $p \in \mathbb{R}^L_{++}$ and define, for all $\gamma \in \mathbb{R}_+$, $S_\gamma = \{x_i \in X \mid p \cdot x_i \leq \gamma\}$. The resulting reference-set mapping is called the **p-money-metric index mapping** and we denote it by w_p. Figure 3 shows the shape of the boundaries of reference sets associated with these two types of indices when $|L|= 2$.

Let us note that homotheticity, which means that boundaries are related by a homothetic transformation (formally, for all $\gamma, \gamma' > 0$, there exists $\mu > 0$ such that, for all $x_i \in X$, $x_i \in \partial S_\gamma \iff \mu x_i \in \partial S_{\gamma'}$), and convexity, are natural additional properties one can impose on reference sets (Piacquadio (2017)). The reference sets associated with ray-indices and money-metric indices satisfy both properties. It is also the case of sets defined by "a concave budget restriction". These sets are convex reference sets of which the boundaries are related by a homotehtic transformation, but are not necessarily lines —see Figure 3.



a. Ray-index reference sets.

b. Money-metric reference sets.

c. "Concave budget restriction" reference sets.

Figure 3. Three examples of homothetic and convex reference sets.

Finally, now that we have formally introduced money-metric indices, we can give an alternative definition of $\lambda^E : X^N \to \mathbb{R}_+$, for $E = (R_N, \Omega) \in \mathcal{E}$ —we refer the reader to

Fleurbaey and Maniquet (2011) (Chapter 2). Let $\Pi^{\Omega} = \{p \in \mathbb{R}^L_{++} \mid p \cdot \Omega = 1\}$. For all $x_N \in X^N$,

$$\lambda^E(x_N) = \max_{p \in \Pi^{\Omega}} \sum_{i \in N} w_p(x_i, R_i). \tag{1}$$

For all $x_N \in X^N$, a price that is solution of (1) is generically denoted by $p^*(x_N, E)$ and called an **E-Scitovsky price** of x_N.

3.3.2 Welfarism

As noted above, reference-set welfare indices are treated as interpersonally comparable representations of individual preferences, measuring the "choice opportunity" of each individual. One can then require that all allocations be compared *only* on the basis of these indices:

Exogenous-reference-set welfarism (Bosmans et al. (2018))

There exist an ordering $\succsim^*$ on $\mathbb{R}^N_+$ and a family of reference sets $S \in \mathcal{S}$ such that, for all $E = (R_N, \Omega) \in \mathcal{D}$, for all $x_N, y_N \in X^N$,

$$x_N \mathbf{R}(E) y_N \iff (w_S(x_i, R_i))_{i \in N} \succsim^* (w_S(y_i, R_i))_{i \in N}. \tag{2}$$

Let $\succ^*$ and $\sim^*$ denote the asymmetric and the symmetric part of $\succsim^*$.

Note that *exogenous-reference-set welfarism* implies that the ranking obtained for an economy (R_N, Ω) does not depend on Ω. Moreover, it implies that the ranking of two allocations only depends on the agents' indifference curves through their bundles at these allocations: the SOF satisfies *unchanged-contour independence*. Here, the term *welfarism* reflects the fact that $\succsim^*$ is defined on vectors of welfare indices and does not depend on the economy under consideration. The term *exogenous* refers to the fact that the family of reference sets does not depend on the economy under consideration either. The ordering and the reference sets are sufficient to compare all allocations: we say that $(\succsim^*, S)$ has **full-scope** to reflect the fact that (2) is satisfied *for all* allocations.

Exogenous-reference-set welfarism is incompatible with the properties we impose:

Proposition 2. *No SOF defined on $\mathcal{E}$ satisfies strong Pareto, continuity, same-preference efficiency-neutral transfer and exogenous-reference-set welfarism.*

This result can be shown by focusing only on Pareto-efficient allocations. Thus, it holds for any measure of efficiency according to which the above transfer axiom may be defined, provided that this measure considers as equivalent any two Pareto-efficient allocations.

Same-preference efficiency-neutral transfer, applied to pairs of Pareto-efficient allocations, and *exogenous-reference-set welfarism* together imply that the associated order $\succsim^*$ is infinitely averse to inequality in welfare indices. Then, the key point is that $(\succsim^*, S)$ has *full-scope*: for any $E = (R_N, \Omega) \in \mathcal{E}$, because for all $x_N, y_N \in X^N$, $x_N \mathbf{R}(E) y_N$ if and only if $w_S^{R_N}(x_N) \succsim^* w_S^{R_N}(y_N)$, the infinite inequality aversion of $\succsim^*$ is, under *strong Pareto*, incompatible with *continuity*, and, under *continuity*, incompatible with *strong Pareto*.

As a consequence, we relax the *full-scope* assumption by requiring that for all $E \in \mathcal{E}$, $(\succsim^*, S)$ be sufficient only for comparing equally efficient allocations.

Assuming the existence of a pair of an ordering and a family of reference sets with *full-scope*, as in *exogenous-reference-set welfarism*, ensures that interpersonal comparisons are made in the same units for all allocations. *Exogenous-reference-set welfarism for same efficiency* relaxes the *full-scope* assumption, which is necessary for the satisfaction of the three other axioms in Proposition 2, and to avoid an infinite aversion to inequality, but, crucially, does so in a way that is *consistent with the restriction of transfers to equally efficient allocations*: for such allocations, interpersonal comparisons are still made in the same units of individual welfare levels.

Exogenous-reference-set welfarism for same efficiency

There exist a monotonic ordering $\succsim^*$ on $\mathbb{R}_+^N$ and a family of reference sets $S \in \mathcal{S}$ such that, for all $E = (R_N, \Omega) \in \mathcal{D}$, for all $x_N, y_N \in X^N$ such that $x_N \mathbf{I}_{\text{eff}}(E) y_N$,

$$x_N \mathbf{R}(E) y_N \iff (w_S(x_i, R_i))_{i \in N} \succsim^* (w_S(y_i, R_i))_{i \in N}. \tag{3}$$

A SOF satisfying *exogenous-reference-set welfarism* satisfies (3). If, in addition, it satisfies *strong Pareto*, then the associated ordering $\succsim^*$ is (strictly) monotonic. We discuss this monotonicity assumption in Section in 7.4 the Appendix. The ordering $\succsim^*$ does not depend on the economy under consideration. It is used to compare *all equally efficient allocations*. Thus, this axiom rules out the comparison of two equally efficient allocations on the indifference set I^1_{eff} according to an ordering $\succsim^1$, and of two equally efficient allocations on the indifference set $I^2_{\text{eff}} \neq I^1_{\text{eff}}$ according to a different ordering $\succsim^2$. The family of reference sets is still the same for all economies.

By imposing *exogenous-reference-set welfarism for same efficiency*, we characterize a large family of SOFs that i) satisfy the first three properties introduced above, in particular our key transfer axiom which applies to Pareto-efficient allocations, and *anonymity*, ii) are compatible with any finite aversion to inequality, except neutrality, and iii) substantially align with the interpersonal comparison principle expressed in *exogenous-reference-set welfarism*.

4 Representation of SOFs with exogenous reference sets

We now state our first representation result. The proof is in Section 7.1 in the Appendix.

4.1 Result

For all $S \in \mathcal{S}$ and $E = (R_N, \Omega) \in \mathcal{E}$, let

$$\bar{D}_S^E = \{(\lambda, w) \in \mathbb{R}_+ \times \mathbb{R}_+^N |\, \lambda^E(x_N) = \lambda,\ w_S^{R_N}(x_N) = w,\ \text{for some } x_N \in X^N\}$$

denote the set of tuples (λ, w) that can be attained in E. For $\Omega \in \mathbb{R}_{++}^L$, let $\bar{D}_S^\Omega = \bigcup_{E=(R_N,\Omega)} \bar{D}_S^E$ denote the set of tuples that can be attained with a preference profile in $\mathcal{R}^N$, given Ω. Finally, let $D_S^E = \{(\lambda, t) \in \mathbb{R}_+ \times \mathbb{R}_+ |\text{there are } i \in N \text{ and } w \in \mathbb{R}_+^N \text{ such that } (\lambda, w) \in \bar{D}_S^E \text{ and } w_i = t\}$, and let $D_S^\Omega = \bigcup_{E=(R_N,\Omega)} D_S^E$.

Theorem 1. *A SOF* $\mathbf{R}$ *defined on* $\mathcal{E}$ *satisfies strong Pareto, continuity, same-preference efficiency-neutral transfer and exogenous-reference-set welfarism for same efficiency, with respect to the family of reference sets* $S \in \mathcal{S}$*, if and only if, for all* $E = (R_N, \Omega) \in \mathcal{E}$*, there is* $F^E : D_S^E \to \mathbb{R}$*, continuous and increasing in each argument, such that, for all* $x_N, y_N \in X^N$*,*

$$\begin{aligned}
& x_N \mathbf{R}(E) y_N \\
\iff & F^E\Big(\lambda^E(x_N), \min_{i\in N} w_S(x_i, R_i)\Big) \geq F^E\Big(\lambda^E(y_N), \min_{i\in N} w_S(y_i, R_i)\Big) \\
\iff & F^E\Big(\max_{p\in\Pi^\Omega} \sum_{i\in N} w_p(x_i, R_i), \min_{i\in N} w_S(x_i, R_i)\Big) \geq F^E\Big(\max_{p\in\Pi^\Omega} \sum_{i\in N} w_p(y_i, R_i), \min_{i\in N} w_S(y_i, R_i)\Big).
\end{aligned} \tag{4}$$

Note that anonymity follows from the axioms of the theorem.

A mean and a min. The first argument of F is the sum of money-metric indices, computed, for each allocation, at any E-Scitovsky price of this allocation. The second argument is the minimal value attained by an exogenous-reference-set index among the agents. Thus, at a superficial level, the functional form in (4) has a familiar structure: comparisons are exclusively based on a sum —equivalently, a mean— and a minimum of "utilities".

In an abstract social choice setting, in which what is ranked are vectors $(u_i)_{i\in N}$ of utility numbers of which the computation is not specified, assumed interpersonally comparable, criteria exclusively based on $\sum_i u_i$ and $\min_i u_i$, have been proposed (Roberts (1980)) and axiomatized (Bossert and Kamaga (2020), Kiguchi and Nakamura (2026)). At the risk of stating the obvious, there are crucial differences with this work. First, we analyze rankings of allocations of multiple physical resources, where agents have heterogeneous preferences, of which (4) involves specific representations. In contrast, the abstract social choice setting is equivalent to the case in which only one good is to be allocated and all agents have the same preference relation. Most importantly, the sum and the minimum in (4) are not computed in the same units —this is due to our relaxation of the *full-scope* assumption, justified by the incompatibility in Proposition 2. Finally, the sum is not *separable*, given the dependence on an endogenous price.

It remains that criteria satisfying (4) are obtained through a monotonic transformation of a sum-functional, neutral toward inequality, and a min-functional, infinitely averse to inequality. In this regard, it is worth mentioning that, while we do not assume it, or derive it in an intermediate step of the proof of Theorem 1, if $\mathbf{R}$ satisfies the axioms, then it has the following property:

Same-preference efficiency-neutral composite transfer

Let $E = (R_N, \Omega) \in \mathcal{D}$ be such that $R_i = R_j = R_k$ for some $i \neq j \neq k \in N$. Let $x_N, y_N \in X^N$ be such that $x_N \mathbf{I}_{\text{eff}}(E) y_N$ and

$$x_h = y_h \text{ for all } h \in N \setminus \{i, j, k\},$$
$$y_k + \delta_2 = x_k > y_i - \delta_1 - \delta_2 = x_i > x_j = y_j + \delta_1, \text{ for some } \delta_1, \delta_2 \in \mathbb{R}^L_{++}.$$

Then, $x_N \mathbf{R}(E) y_N$.

This property is reminiscent of the key axiom assumed in Bossert and Kamaga (2020) for abstract utilities, and states that a composition of progressive and regressive rank-preserving transfers, which involves three individuals with the same preference relation, and preserves efficiency, is not socially considered as a regression.

In this work, this property is a consequence of the infinite aversion to inequality of $\succsim^*$, which follows from the conjunction of *same-preference efficiency-neutral transfer* and *exogenous-reference-set welfarism for same efficiency* (see Lemma 2). Of course, whether *same-preference efficiency-neutral composite transfer* is a convincing fairness principle, when assumed in addition to *same-preference efficiency-neutral transfer*, is debatable.[24] On the other hand, as highlighted in the introduction, a series of results consist in showing that, combined with *strong Pareto* and a given "mild" transfer axiom, some informational parsimony requirement turns out to imply a "radical" transfer property. One example is the com-

[24]These properties are logically independent.

bination of *strong Pareto*, *same-preference transfer* and *unchanged-contour independence*, which implies *same-preference priority* (Fleurbaey and Maniquet (2011)).[25] Another example is the combination, at play in the impossibility stated in Proposition 2, of *strong Pareto*, *same-preference efficiency-neutral transfer* and the *exogenous-reference-set welfarism* axiom (which is stronger than *unchanged-contour independence*), adopted in Bosmans et al. (2018). The combination of the three axioms also implies *same-preference priority.* Theorem 1, in which the informational axiom is *exogenous-reference-set welfarism for same efficiency*, is in the vein of these results, with the fundamental difference that the implied property of composite transfer is still compatible with *any* inequality aversion.

Furthermore, *strong Pareto* and *exogenous-reference-set welfarism for same efficiency* merely preclude infinite aversion and neutrality toward inequality, respectively.

Who is the worst-off? The above representation holds for any family $S \in \mathcal{S}$, thus allowing considerable freedom in selecting the index measuring welfare disparities among agents, *i.e.* in deciding who is the worst-off of any two agents whose preference relations differ. This flexibility is consistent with recent axiomatic approaches to the derivation of individual welfare indices, which deliberately abstract from the precise manner in which these indices are "later" aggregated to construct a ranking of allocations —Fleurbaey and Maniquet (2017b, 2018), Decancq and Nys (2021), Maniquet and Moramarco (2022, 2025), Capeau et al. (2024)).[26]

This is in contrast with the representation derived in Bosmans et al. (2018), for which the axioms single-out the use of exogenous money-metric indices, as arguments of a Schur-concave monotonic function.[27]

Therefore, Theorem 1 shows that by allowing a SOF to depend on the social endowment, and weakening the *full-scope* assumption imposed on exogenous-reference-set indices, one is able to rank allocations consistently with a stronger transfer principle (covering the fundamental case of Pareto-efficient allocations), and to preserve the latitude to choose —in light of other normative requirements— among all exogenous-reference-set indices to conduct

[25]Here is a definition of *same-preference priority.* Let $E = (R_N, \Omega) \in \mathcal{D}$ be such that $R_i = R_j$ for some $i, j \in N$. Let $x_N, y_N \in X^N$ be such that $x_k = y_k$ for all $k \in N \setminus \{i, j\}$, and $y_i > x_i > x_j > y_j$. Then, $x_N \mathbf{R}(E) y_N$.

[26]See the addendum to the first paper cited here (Fleurbaey and Maniquet (2017a)).

[27]Their theorem can be stated, in the formalism of SOFs, in the following way. A SOF $\mathbf{R}$ defined on $\mathcal{E}$ satisfies *strong Pareto*, *anonymity*, *continuity*, *independence from available quantities*, *same-preference transfer for allocations with same Scitovsky set* and *exogenous-reference-set welfarism*, with respect to the family of reference sets $S \in \mathcal{S}$, if and only if there is $p \in \mathbb{R}^L_{++}$ such that, for all $\gamma \in \mathbb{R}_+$, $S_\gamma = \{x_i \in X \mid p \cdot x_i \leq \gamma\}$, and there is $F : \mathbb{R}^N_+ \to \mathbb{R}$, continuous, Schur-concave and increasing in each argument, such that, and all $E = (R_N, \Omega) \in \mathcal{E}$, for all $x_N, y_N \in X^N$, $x_N \mathbf{R}(E) y_N \iff F((w_p(x_i, R_i))_{i \in N}) \geq F((w_p(y_i, R_i))_{i \in N})$.

interpersonal comparisons.[28]

Although we see this flexibility as an asset, results that offer strong guidance regarding the choice of individual welfare indices, like the one in Bosmans et al. (2018), valuably complement our general representation result: in Section 5, we characterize a unique measure, among a family that contains all exogenous-reference-set indices.

4.2 Further discussion

4.2.1 Unchanged-contour independence

In Theorem 1, the SOF $\mathbf{R}$ is not assumed to satisfy unchanged-contour independence. It should be noted that the dependence of the social ordering on the considered preference profile is already constrained by the fact that, at each allocation, the values of the measure of efficiency and the welfare index only depend on the associated indifference curves. Still, the mapping F in (4) may differ accross economies.[29]

It is easy to find SOFs satisfying the axioms of Theorem 1 and *unchanged-contour independence*: let $\mathbf{R}$ be a SOF defined on $\mathcal{E}$ such that there exist a family of reference sets $S \in \mathcal{S}$, such that, for all $\Omega \in \mathbb{R}^L_{++}$, there is $F^\Omega : D^\Omega_S \to \mathbb{R}$, continuous and increasing in each argument, such that, for all $E = (R_N, \Omega) \in \mathcal{E}$, and all $x_N, y_N \in X^N$,

$$x_N \mathbf{R}(E) y_N \iff F^\Omega\Big(\lambda^E(x_N), \min_{i \in N} w_S(x_i, R_i)\Big) \geq F^\Omega\Big(\lambda^E(y_N), \min_{i \in N} w_S(y_i, R_i)\Big).$$

Whether the axioms of Theorem 1 and *unchanged-contour independence* actually imply such a representation remains an open question.

4.2.2 From maximin to leximin

One may argue that fostering equity, conditional on efficiency, requires that a same-preference transfer which relates two equally efficient allocations be considered a strict improvement. This corresponds to the replacement, in the conclusion of our transfer axiom, of $\mathbf{R}$ by $\mathbf{P}$. Let us call the resulting axiom **Strong same-preference efficiency-neutral transfer**. From now, we use the adjective "strong" for any axiom defined by replacing weak preference by strict preference in the conclusion of another axiom. Lemma 2 in the proof of Theorem 1 states that $\succsim^*$, defined on vectors of reference-set welfare indices, satisfies *Hammond equity*

[28]SOFs that additionally satisfy *unchanged-contour independence* also preserve this latitude (see Section 4.2.1).

[29]We allow ourselves to be informal here: a SOF of which the image for $E = (R_N, \Omega)$ is represented by F_L if all R_i is linear, and by F_0 otherwise —where $F_L \neq F_0$ have all the properties stated in Theorem 1— satisfy our axioms, but violate *unchanged-contour independence*.

(Hammond (1976)).[30] Replacing *same-preference efficiency-neutral transfer* by *strong same-preference efficiency-neutral transfer* immediately yields *strong Hammond equity*. Then, as is well-known, $\succsim^*$ cannot be continuous on $\mathbb{R}^N_+$. Let us present a family of criteria satisfying *strong same-preference efficiency-neutral transfer*.

Let

$$\hat{D}^E_S = \{(\frac{\lambda}{|N|}, t) \in \mathbb{R}_+ \times \mathbb{R}_+ | \text{there are } i \in N \text{ and } w \in \mathbb{R}^N_+ \text{ such that } (\lambda, w) \in \bar{D}^E_S \text{ and } w_i = t\}.$$

Let $\mathbf{R}$ be a SOF defined on $\mathcal{E}$ such that there is $S \in \mathcal{S}$, such that, for all $E = (R_N, \Omega) \in \mathcal{E}$, there is $f^E : \hat{D}^E_S \to \mathbb{R}$, continuous and increasing in each argument, such that, for all $x_N, y_N \in X^N$,

$$\begin{aligned} & x_N \mathbf{R}(E) y_N \\ \iff & \Big[f^E\Big(\frac{\lambda^E(x_N)}{|N|}, w_S(x_i, R_i)\Big)\Big]_{i \in N} \geq_{\text{lex}} \Big[f^E\Big(\frac{\lambda^E(y_N)}{|N|}, w_S(y_i, R_i)\Big)\Big]_{i \in N}, \end{aligned} \tag{5}$$

where $\geq_{\text{lex}}$ denotes the leximin ordering on $\mathbb{R}^N$.[31]

Importantly, in spite of the use of the leximin ordering on $\mathbb{R}^N_+$, in each $E \in \mathcal{E}$, $\mathbf{R}(E)$ may display, depending on f^E, any finite aversion to inequality, except neutrality.

The SOF $\mathbf{R}$ satisfies *anonymity*, *strong same-preference efficiency-neutral transfer* and, importantly, all the other axioms of Theorem 1, except *continuity*.

In addition, $\mathbf{R}$ satisfies *strong same-preference efficiency-neutral composite transfer*. Similarly to the discussion above, which we do not repeat, this property, which is a consequence of our axioms, is reminiscent of the key axiom *assumed* in Kamaga (2018) for abstract utilities.

4.3 An alternative equity axiom

Same-preference efficiency-neutral transfer might be objected to on the same ground as any "Pigou-Dalton axiom": as Amiel and Cowell (1999) put it when they propose an interpretation for the fact that a majority of their respondents reject the Pigou-Dalton principle, one

[30] Let us recall the definition of *Hammond equity*. Let $u, v \in \mathbb{R}^N_+$ be such that $v_i < u_i < u_j < v_j$, for some $i, j \in N$, and, and $u_k = v_k$ for all $k \in N \setminus \{i, j\}$. Then, $u \succsim^* v$.

[31] Let us recall the definition of the leximin ordering. For any $u \in \mathbb{R}^N$, let $u_\uparrow$ denote the vector obtained by ordering the components of u in non-decreasing order. The ordering $\geq_{\text{lex}}$ is defined by: $u \geq_{\text{lex}} v$ if and only if

$$\Big[[u_{\uparrow_1} > v_{\uparrow_1}] \text{ or } [u_{\uparrow_1} = v_{\uparrow_1} \text{ and } u_{\uparrow_2} > v_{\uparrow_2}] \text{ or } \ldots\ [u_{\uparrow_k} = v_{\uparrow_k} \text{ for all } k = 1, \ldots, N{-}1 \text{ and } u_{\uparrow_{|N|}} > v_{\uparrow_{|N|}}] \text{ or } [u = v]\Big].$$

might be "*concerned about the overall structure of income differences and not just about the incomes of the particular individuals who are involved in the transfer*" —they accordingly observe that transfers among agents "in the middle" lead to substantially more rejection than transfers from the richest to the poorest agents.[32] We introduce here an axiom which is immune to this objection, as it is restricted to situations in which inequality is unambiguously reduced, while efficiency is preserved.

Same-preference efficiency-neutral uniform inequality reduction (with respect to $\mathbf{R}_{\text{eff}}$)

Let $E = (R_N, \Omega) \in \mathcal{D}$ be such that, for all $i \in N$, $R_i = R_0$. Let $x_N, y_N \in X^N$ be such that $x_N \mathbf{I}_{\text{eff}}(E) y_N$ and there is $j \in N$ such that, for some $\Delta \in \mathbb{R}^L_{++}$,

$$\begin{aligned} &y_k = y_{k'}, \text{ for all } k, k' \in N \setminus \{j\}, \\ &x_k = y_k - \Delta > x_j = y_j + (|N|-1)\Delta, \text{ for all } k \in N \setminus \{j\}. \end{aligned}$$

Then, $x_N \mathbf{R}(E) y_N$.

In contrast to our previous transfer axiom, in *same-preference efficiency-neutral uniform inequality reduction*, all agents are affected by the change from y_N to x_N. Because they all have the same preference relation, the properties of x_N and y_N imply that agent j is, unambiguously, the unique worst-off agent in both allocations, and that the other agents are equally well-off in x_N, and equally well-off in y_N.[33]

A simple modification of the proof of Theorem 1 —included, for completeness, in Section 7.1.3 in the Appendix— shows that the conclusion of Theorem 1 holds if we replace our transfer axiom by the above axiom. Under *strong Pareto*, *continuity* and *exogenous-reference-set welfarism for same efficiency*, the two equity axioms are equivalent.

Theorem 2. *A SOF* $\mathbf{R}$ *defined on* $\mathcal{E}$ *satisfies strong Pareto, continuity, same-preference efficiency-neutral uniform inequality reduction and exogenous-reference-set welfarism for same efficiency, with respect to the family of reference sets* $S \in \mathcal{S}$*, if and only if, for all* $E = (R_N, \Omega) \in \mathcal{E}$*, there is* $F^E : D^E_S \to \mathbb{R}$*, continuous and increasing in each argument, such that, for all* $x_N, y_N \in X^N$*,*

$$x_N \mathbf{R}(E) y_N \iff F^E\Big(\lambda^E(x_N), \min_{i\in N} w_S(x_i, R_i)\Big) \geq F^E\Big(\lambda^E(y_N), \min_{i\in N} w_S(y_i, R_i)\Big).$$

[32] We refer the reader to the discussion in pages 45-72 in Amiel and Cowell (1999).

[33] In the context of abstract social choice, the restriction to situations where one agent is the unique worst-off agent in both utility vectors, and the other agents are equally well-off in the first vector, and equally well-off in the second one, is studied by Bosmans and Ooghe (2013). They study transfers of utility that may be leaky.

Note that anonymity follows from the axioms of the theorem.

5 Representation of SOFs with resource-dependent reference sets

Considering the dependence on available resources appearing in the definition of a SOF, and endorsed in a core motivation of this work, according to which progressive transfers relating Pareto-efficient allocations should be considered socially desirable, it is legitimate to allow reference sets to depend on the social endowment Ω.[34]

We refer to a collection assigning a family of reference sets to each Ω as a **system of families of reference sets**, and we denote it by $\mathbf{S} : \Omega \in \mathbb{R}^L_{++} \mapsto S^\Omega \in \mathcal{S}$.

In Section 5.2, we show that a result analogous to Theorem 1 and Theorem 2 holds when one considers systems of families of reference sets rather than exogenous families of reference sets.

In Section 5.3, we single-out a well-known system of families of reference sets. The core axiom under consideration is a transfer axiom that drops the same-preference assumption and emphasizes an interpretation of equity as resource equality, while it remains compatible with strong Pareto —it is thus weaker than transfer.

5.1 Resource-dependent reference sets

The two prominent types of indices induced by a system of families of reference sets take the form of special ray-indices and money-metric indices. For $S^\Omega_\gamma = \{x_i \in X \mid x_i \leqq \gamma\Omega\}$, $\gamma \in \mathbb{R}_+$, the family of reference sets is called the **Ω-ray family** and the resulting mapping is called the **Ω-ray index mapping**, denoted by w_Ω. With an abuse of language, we will use the name **Ω-ray system** to refer to the system that associates each Ω with the Ω-ray family. Let $\mathbf{p} : \Omega \in \mathbb{R}^L_{++} \mapsto p^\Omega \in \mathbb{R}^L_{++}$ and define $S^\Omega_\gamma = \left\{x_i \in X \mid p^\Omega \cdot x_i \leq \gamma\right\}$, for all $\gamma \in \mathbb{R}_+$. The family of reference set is called the **p^Ω-family**, and the resulting reference-set mapping is called the **p^Ω-money-metric index mapping**, denoted by w_{p^Ω}. For each mapping $\mathbf{p}$, the induced system of family of reference sets is called the **$\mathbf{p}$-system**.

The number $w_\Omega(x_i, R_i)$ is the portion of the social endowment that makes agent i with preference R_i indifferent to x_i. The number $w_{p^\Omega}(x_i, R_i)$ is the minimum income, computed at a reference price —the choice of which may depend on Ω, for example a price reflecting

[34] The SOFs in Theorems 1 and 2 must depend on Ω, through λ^E, and may additionally depend on Ω through F^E.

the relative available quantities— that she needs in order to reach a bundle that she finds at least as desirable as x_i.

We adopt the following weakening of *exogenous-reference-set welfarism for same efficiency*:

Ω-reference-set welfarism for same efficiency

There exist a monotonic ordering $\succsim^*$ on $\mathbb{R}^N_+$ and a system of families of reference sets $\mathbf{S}$ such that, for all $E = (R_N, \Omega) \in \mathcal{D}$, for all $x_N, y_N \in X^N$ such that $x_N \mathbf{I}_{\text{eff}}(E) y_N$,

$$x_N \mathbf{R}(E) y_N \iff (w_{S^\Omega}(x_i, R_i))_{i \in N} \succsim^* (w_{S^\Omega}(y_i, R_i))_{i \in N}.$$

5.2 General representation

That the ranking criterion is obtained on the basis of two separate evaluations, a maximin one, and an inequality-neutral one measuring efficiency, as in Theorem 1, also follows when *exogenous-reference-set welfarism for same efficiency* is replaced by Ω*-reference-set welfarism for same efficiency*, for a wide class of systems of families of reference sets.

In this section, we assume that the systems have the following technical property. It puts constraint on the shape of the reference sets used for each $\Omega \in \mathbb{R}^L_{++}$. We note that this is a condition under which we can prove the representation in (6) below, but it is not necessary for (6) to hold, as we show in Section 5.3.

For all $E = (R_N, \Omega) \in \mathcal{E}$, let $\text{Prop}(E)$ denote the set of allocations that are proportional to Ω, *i.e.*, $\text{Prop}(E) = \{x_N \in X^N \mid x_i = \mu_i \Omega \text{ for some } \mu_i \in \mathbb{R}_+, \text{ for all } i \in N\}$. In such cases, we also say that the bundle x_i is proportional (to Ω), and if $x_i = \mu_i \Omega > y_j = \mu_j \Omega$, we say that x_i is higher than y_j.

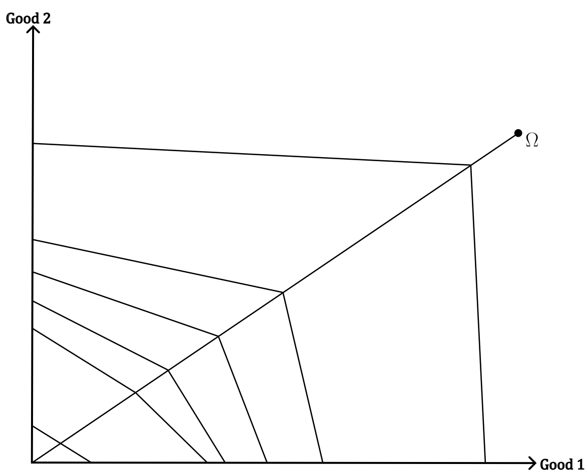
Before stating the property, it is worth mentioning, in order to facilitate the interpretation, that if a reference set S_γ, for $\gamma > 0$, is strictly lower-comprehensive, then, for all $t > 0$, for all $z_i \in \partial S_\gamma$, there are $l \in L$ and $\bar{z}_i \in B(z_i, t) \cap \partial S_\gamma$ such that $\bar{z}^l_i - z^l_i > 0$.

We do not restrict to systems of families of strictly lower-comprehensive reference sets, but we impose a property which is close in spirit. It states that for all $\Omega \in \mathbb{R}^L_{++}$, given a parameter $t > 0$, there is $\theta > 0$ such that, for all proportional bundle z_i, for all open ball around z_i of radius at least t, one can find a bundle $\bar{z}_i$ on the same reference-set-boundary, $\partial S^\Omega(z_i)$, of which the l-coordinate is higher, by at least θ, than the l-coordinate of z_i:

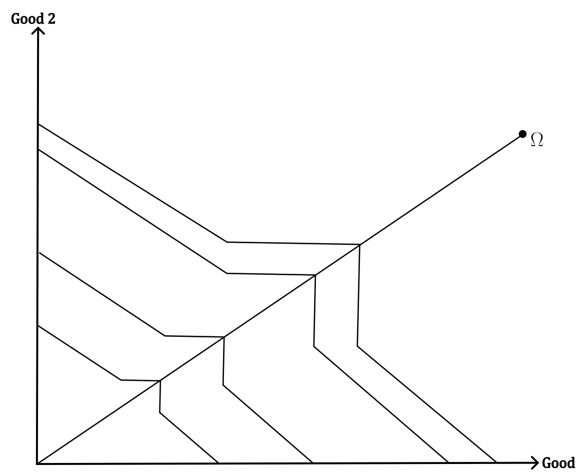
Lower bound on lower-comprehensiveness (around proportional bundles)

For all $\Omega \in \mathbb{R}^L_{++}$, for all $t > 0$, there is $\theta > 0$ such that, for all $z_i \in \text{ray}[0, \Omega) \setminus \{\mathbf{0}\}$, and all $t' \geq t$, there are $l \in L$ and $\bar{z}_i \in B(z_i, t') \cap \partial S^{\Omega}(z_i)$ such that $\bar{z}_i^l \geq z_i^l + \theta$.[35]

For any mapping $\mathbf{p} : \mathbb{R}^L_{++} \to \mathbb{R}^L_{++}$, the $\mathbf{p}$-system satisfies the *lower bound on lower-comprehensiveness* property. The Ω-ray system does not satisfy it —let us mention that we actually provide a characterization of the Ω-ray system in Section 5.3. In Figure 4, fixing Ω, we give two additional examples of violations of this property, drawing only the boundaries of some references sets. In the first case, the reference sets are strictly lower-comprehensive around any point on ray$[\mathbf{0}, \Omega)$. However, as γ tends to infinity, they converge (for the Hausdorff distance) to a reference set which is strictly-lower comprehensive around *none* of the points of its boundary. In the second case, in which reference sets are homothetic, for any point z_i on ray$[\mathbf{0}, \Omega)$, there is a neighborhood of z_i such that $S^{\Omega}(z_i)$ is strictly-lower comprehensive around *none* of the points of the intersection of its boundary and this neighborhood.



Case 1.



Case 2.

Figure 4. Two examples of violations of *lower bound on lower-comprehensiveness*.

We can now state our representation result. The proof is in Section 7.2 in the Appendix.

Theorem 3. *A SOF* $\mathbf{R}$ *defined on* $\mathcal{E}$ *satisfies strong Pareto, continuity,* [*same-preference efficiency-neutral transfer* or *same-preference efficiency-neutral uniform inequality reduction*] *and Ω-reference-set welfarism for same efficiency, with respect to the system of families of reference sets* $\mathbf{S}$ *that has the lower bound on lower-comprehensiveness property, if and only*

[35]Given the restriction to proportional bundles, this property does not imply that all reference sets are strictly lower comprehensive.

if, for all $E = (R_N, \Omega) \in \mathcal{E}$, there is $F^E : D^E_{S^\Omega} \to \mathbb{R}$, continuous and increasing in each argument, such that, for all $x_N, y_N \in X^N$,

$$\begin{aligned}
& x_N \mathbf{R}(E) y_N \\
\iff & F^E\Big(\lambda^E(x_N), \min_{i\in N} w_{S^\Omega}(x_i, R_i)\Big) \geq F^E\Big(\lambda^E(y_N), \min_{i\in N} w_{S^\Omega}(y_i, R_i)\Big) \\
\iff & F^E\Big(\max_{p\in\Pi^\Omega} \sum_{i\in N} w_p(x_i, R_i), \min_{i\in N} w_{S^\Omega}(x_i, R_i)\Big) \geq F^E\Big(\max_{p\in\Pi^\Omega} \sum_{i\in N} w_p(y_i, R_i), \min_{i\in N} w_{S^\Omega}(y_i, R_i)\Big).
\end{aligned} \tag{6}$$

Note that anonymity follows from the axioms of the theorem.

5.3 Characterization of SOFs based on the Ω-ray system

The SOFs that we have characterized so far offer a lot of flexibility with respect to individual welfare indices used to compare equally efficient allocations. In this section, we single-out a system of families of reference sets, mainly on the basis of a transfer axiom that drops the same-preference assumption, but restricts attention, in each economy, to a specific region of X^N.

This axiom, as compared to *same-preference efficiency-neutral transfer*, emphasizes an interpretation of equity as *resource equality*, while it remains compatible with *strong Pareto*. It is inspired by the equity property due to Steinhaus (1948) for the problem of selecting allocations. The underlying view is that in all $E = (R_N, \Omega) \in \mathcal{E}$, an equal division of Ω would be equitable, although generally inefficient, and that, as a consequence, each agent should be given a bundle that she weakly prefers to $\frac{\Omega}{|N|}$.

For the problem of ranking allocations, the importance of *equal-division as a lower-bound*, expressing a compromise between a notion of fairness expressed in physical terms, and a preference-based efficiency requirement, can be captured in the following axiom. The idea is that if no feasible Pareto-improvement can be made starting from the equal-division allocation, which realizes the objective of resource equality, then there is no reason to socially prefer a feasible allocation to the equal-division allocation.

Selection of Pareto-efficient equal-division

Let $E = (R_N, \Omega) \in \mathcal{D}$ be such that $x_N = (\frac{\Omega}{|N|}, \cdots, \frac{\Omega}{|N|}) \in \text{Par}(E)$. Then, $x_N \in \max|_{\mathbf{R}(E)} \text{Fea}(E)$.

It comes as no surprise that any SOF satisfying the *welfarism* axiom adopted in Bosmans et al. (2018) —*exogenous-reference-set welfarism*— such as a SOF comparing allocations on the basis of money-metric indices computed from an exogenous price, violates this property.

We propose a transfer axiom in which the equal-division bundle is a benchmark, in the sense that an individual who has more of every good than what she would receive under equal-division is considered better-off than an individual who has less, even if their indifference curves, at the considered bundles, cross. Fleurbaey and Maniquet (2011) proposed a transfer axiom in which the equal-division bundle plays this role. However, in contrast to our axiom, it makes no reference to the respective efficiency of the pre-and-post-transfer allocations, ignoring the potential trade-off between efficiency and equity when evaluating a transfer.

Our axiom, arguably, better reflects the efficiency motivation behind *selection of Pareto-efficient equal-division*, since the underlying view is that the only reason to depart from resource equality pertains to the presence of efficiency gains.

Equal-division efficiency-neutral transfer (with respect to $\mathbf{R}_{\text{eff}}$)
Let $E = (R_N, \Omega) \in \mathcal{D}$. Let $x_N, y_N \in X^N$ be such that $x_N \mathbf{I}_{\text{eff}} y_N$ and, for some $i \neq j \in N$,

$$x_k = y_k \text{ for all } k \in N \setminus \{i, j\},$$
$$y_i - \Delta = x_i > \frac{\Omega}{|N|} > x_j = y_j + \Delta, \text{ for some } \Delta \in \mathbb{R}^L_{++}.$$

Then, $x_N \mathbf{R}(E) y_N$. In addition, if, for all $k \in N \setminus \{j\}$, $y_j < y_k$ and $x_j < x_k$, then $x_N \mathbf{P}(E) y_N$.

If a transfer of positive quantities of each good is made from i to j such that i) all the other agents are unaffected, ii) after the transfer, i still consumes more than her equal-division share, while j still consumes less, and iii) this transfer does not affect efficiency, then the transfer is socially considered as a weak improvement. If in addition, in both of the pre-and-post-transfer allocations, j is the agent who receives the least, then the transfer is socially considered as an improvement.

We study *equal-division efficiency-neutral transfer* in conjunction with our three basic axioms, with *Ω-reference-set welfarism for same efficiency*, and *independence from proportional expansions* —which, as a recall, imposes that a SOF depend on the social endowment only through the *relative* scarcity of resources. These axioms together imply that, in each economy, two equally efficient allocations are compared according to the welfare level, measured by the Ω-ray index mapping, of the worst-off agent(s):

Theorem 4. *A SOF* $\mathbf{R}$ *defined on* $\mathcal{E}$ *satisfies strong Pareto, continuity, independence from proportional expansions, equal-division efficiency-neutral transfer and Ω-reference-set welfarism for same efficiency, with respect to the system of families of reference sets* $\mathbf{S}$*, if and only if* $\mathbf{S}$ *is the Ω-ray system and, for all $E = (R_N, \Omega) \in \mathcal{E}$, there is $F^E : D^E_{S\Omega} \to \mathbb{R}$,* *continuous and increasing in each argument, such that, for all* $x_N, y_N \in X^N$*,*

$$\begin{aligned} & x_N \mathbf{R}(E) y_N \\ \iff & F^E(\lambda^E(x_N), \min_{i \in N} w_\Omega(x_i, R_i)) \geq F^E(\lambda^E(y_N), \min_{i \in N} w_\Omega(y_i, R_i)) \\ \iff & F^E\Big(\max_{p \in \Pi^\Omega} \sum_{i \in N} w_p(x_i, R_i), \min_{i \in N} w_\Omega(x_i, R_i)\Big) \geq F^E\Big(\max_{p \in \Pi^\Omega} \sum_{i \in N} w_p(y_i, R_i), \min_{i \in N} w_\Omega(y_i, R_i)\Big), \end{aligned}$$

and, for all $\mu \in \mathbb{R}_+$ *and* $E^\mu = (R_N, \mu\Omega) \in \mathcal{E}$*,* F^E *is an increasing transformation of* F^{E^μ}*.*

Note that anonymity follows from the axioms of the theorem.

The proof is in Section 7.3 in the Appendix.

Let $\mathbf{R}$ be as in Theorem 4. Then, $\mathbf{R}$ additionally satisfies *same-preference efficiency-neutral transfer* and *same-preference efficiency-neutral uniform inequality reduction.*

Importantly, $\mathbf{R}$ *rationalizes* the **Ω-egalitarian-equivalent correspondence**, which is one of the two main solutions proposed for the problem of selecting "fair" allocations —see Thomson (2011) for a review.[36] For all $E = (R_N, \Omega) \in \mathcal{E}$, this allocation rule selects the subset of feasible allocations

$$\mathrm{EgEq}(E) = \{z_N \in \mathrm{Par}(E) \mid \text{ there is } \gamma \in \mathbb{R}_+ \text{ such that, for all } i \in N,\ z_i I_i \gamma\Omega\}.$$

Then, for all $E = (R_N, \Omega) \in \mathcal{E}$, $\max|_{\mathbf{R}(E)} \mathrm{Fea}(E) = \mathrm{EgEq}(E)$.

6 Conclusion

We analyzed the construction of Social Ordering Functions (SOFs), which map each economy —each pair of a preference profile and a social endowment of resources to be divided among equally entitled agents— to rankings of allocations. Starting from the fact that the conjunction of strong Pareto, same-preference transfer and unchanged-contour independence forces society to give absolute priority to the worst-off, we centered our analysis around a new fairness principle, restricting attention to efficiency-neutral transfers, building upon the case made by Bosmans et al. (2018).

We provided several axiomatization results in which the characterized SOFs may display *any* aversion to inequality, except neutrality and absolute priority to the worst-off. All

[36] The origins of this allocation rule trace back to Kolm (1969) and Pazner and Schmeidler (1978). The other main solution is the *egalitarian Walrasian correspondence.* Informally, this rule selects, in each economy, the allocations associated with the competitive equilibria starting from equal-division as the vector of initial endowments. These allocations are Pareto-efficient and satisfy the fairness property of *no-envy* (Foley (1967), Varian (1974)). We briefly discuss this allocation rule and SOFs that rationalize it in Section 6, which concludes the paper.

these SOFs must have the following property: there is a reference-set mapping of welfare indices, such that, for two equally efficient allocations, one is socially at least as desirable as the other one if and only if the worst-off individuals, according to this reference-set measure, are better-off in this allocation than in the other one. We defined same-preference efficiency-neutral transfer, as well as same-preference efficiency-neutral uniform inequality reduction, aiming at applying a redistribution principle to the *Pareto-efficient allocations* of an economy, as well as to equally inefficient allocations. The SOFs we first characterized (in Sections 4 and 5.2) on the basis of these axioms preserve the latitude to choose —in light of additional normative requirements— among a vast family of reference-set indices to conduct interpersonal comparisons. Then, we characterized a unique reference-set measure, relying on a transfer axiom which, conditional on the efficiency of the pre-and-post-transfer allocations, gives special importance to the *equal-division lower bound*. The resulting SOFs may still display any finite aversion to inequality, except neutrality, and they all rationalize the Ω-*egalitarian-equivalent correspondence*, which is central to the theory of fair allocation.

This last result naturally raises the issue of identifying SOFs defined on $\mathcal{E}$ that i) satisfy *strong Pareto* and *same-preference efficiency-neutral transfer*, ii) do not give absolute priority to the worst-off and iii) rationalize the other central solution to the theory of fair allocation, namely the *egalitarian Walrasian correspondence*. The following SOFs have these properties, and, additionally, satisfy *continuity*, *anonymity*, and *unchanged-contour independence*. These are SOFs $\mathbf{R}$ defined on $\mathcal{E}$ such that, for all $\Omega \in \mathbb{R}^L_{++}$, there is $F^\Omega : \mathbb{R}^2_+ \to \mathbb{R}$, continuous and increasing in each argument, such that, for all $E = (R_N, \Omega) \in \mathcal{E}$, and all $x_N, y_N \in X^N$,

$$x_N \mathbf{R}(E) y_N \iff F^\Omega\Big(\lambda^E(x_N), \max_{p \in \Pi^\Omega} \min_{i \in N} w_p(x_i, R_i)\Big) \geq F^\Omega\Big(\lambda^E(y_N), \max_{p \in \Pi^\Omega} \min_{i \in N} w_p(y_i, R_i)\Big).$$

For these SOFs, equally efficient allocations are compared on the basis of vectors of money-metric welfare indices that are *endogenous and non-separable*. That is, the price used to compute the money-metric welfare level of each individual does not merely depend on her bundle and her indifference curve at this bundle, but also on those of the other agents. The fact that, for two equally efficient allocations, interpersonal comparisons are not made in the same units, is a violation of Ω-*reference-set welfarism for same efficiency*. From a technical perspective, since we work with a weaker axiom than *same-preference transfer*, this feature substantially complicates the analysis. Characterizing —among the SOFs that compare two equally efficient allocations on the basis on an ordering $\succsim^*$ defined on vectors of endogenous and non-separable money-metric indices— the SOFs that satisfy the axioms of Theorem 1, except *reference-set welfarism for same efficiency*, is a highly challenging task.

7 Appendix

7.1 Proof of Theorem 1

The *sufficiency part* is readily checked, and we only prove the *necessity part.*

Lemma 1. *Let* $\mathbf{R}$ *be a SOF defined on* $\mathcal{E}$ *satisfying continuity and exogenous-reference-set welfarism for same efficiency, with respect to the family of reference sets* $S \in \mathcal{S}$*. Let* $E = (R_N, \Omega) \in \mathcal{E}$*. There exists a continuous mapping* $f : \bar{D}_S^E \to \mathbb{R}$ *such that, for all* $x_N, y_N \in X^N$*,*

$$x_N \mathbf{R}(E) y_N \iff f\Big(\lambda^E(x_N), w_S^{R_N}(x_N)\Big) \geq f\Big(\lambda^E(y_N), w_S^{R_N}(y_N)\Big).$$

Proof. Let $E = (R_N, \Omega) \in \mathcal{E}$. As $\mathbf{R}(E)$ is continuous on X^N and w_S is continuous on X, *exogenous-reference-set welfarism for same efficiency* implies that $\succsim^*$ is continuous on $\mathbb{R}_+^N$. Thus, there exists a continuous mapping $W : \mathbb{R}_+^N \to \mathbb{R}$ representing $\succsim^*$. There also exists a continuous mapping $\tilde{f} : X^N \to \mathbb{R}$ representing $\mathbf{R}(E)$.

Let $x_N, y_N \in X^N$ be such that $\lambda^E(x_N) = \lambda^E(y_N) = \lambda \geq 0$. Then,

$$\begin{aligned} & x_N \mathbf{R}(E) y_N \\ \iff & W(w_S^{R_N}(x_N)) \geq W(w_S^{R_N}(y_N)) \\ \iff & \tilde{f}(x_N) \geq \tilde{f}(y_N). \end{aligned}$$

As a consequence, there is a mapping $g^\lambda : W(\lambda^{-1}) \to \mathbb{R}$, where

$$W(\lambda^{-1}) = \{t \in \mathbb{R} | \text{ there is } z_N \in X^N \text{ such that } \lambda^E(z_N) = \lambda \text{ and } W(w_S^{R_N}(z_N)) = t\},$$

continuous and increasing, such that, for all $x_N \in X^N$ such that $\lambda^E(x_N) = \lambda$,

$$\tilde{f}(x_N) = g^\lambda\Big(W(w_S^{R_N}(x_N))\Big).$$

Defining

$$\begin{aligned} f : \bar{D}_S^E &\to \mathbb{R} \\ (\lambda, w) &\mapsto g^\lambda(W(w)) \end{aligned}$$

yields the result. $\square$

Lemma 2. *Let* $\mathbf{R}$ *be a SOF defined on* $\mathcal{E}$ *satisfying same-preference efficiency-neutral transfer and exogenous-reference-set welfarism for same efficiency, with respect to the family of reference sets* $S \in \mathcal{S}$*. Then, the associated ordering* $\succsim^*$ *satisfies Hammond equity: let* $u, v \in \mathbb{R}_+^N$ *be such that, for some* $i, j \in N$,

$$v_i < u_i < u_j < v_j, \text{ and,}$$
$$u_k = v_k \text{ for all } k \in N \setminus \{i, j\},$$

then, $u \succsim^* v$.

Proof. Some notation must be introduced first. For all $u \in \mathbb{R}_+^N$, and all $k \in N$, S_{u_k} is the reference sets in S of which the index is u_k. For any $x_i \in X$ and any $\gamma \in \mathbb{R}_+$, let $s_\gamma(x_i) = \{y_i \in \partial S_\gamma \mid \neg[x_i > y_i]\}$ be the set of bundles *on the boundary* of S_γ that are not strictly dominated by x_i. This set may be empty. However, for all $\gamma > \gamma'$, because $S_{\gamma'} \subsetneq S_\gamma$ and S_γ is comprehensive, there is $x_i \in \partial S_\gamma$ such that $s_{\gamma'}(x_i)$ is non-empty. We say that z_i *is above* S_γ if there is $z_i' \in \partial S_\gamma$ such that $z_i > z_i'$.

Let $y_j \in \partial S_{v_j}$ be such that $s_{v_i}(y_j) \neq \emptyset$. Without loss of generality, we assume that $y_j > \mathbf{0}$. The construction is illustrated in Figure 5, for the case in which $|L|= 2$, in which we only represent the indifference curves of agents i and j.

Let $l \in L$. The choice of y_j implies that there are $\mathbf{0} \leqq y_i < y_j$ (close enough to y_j), above S_{v_i}, and $0 < \alpha < \frac{1}{2}$ (close enough to $\frac{1}{2}$), such that

i) $x_j = y_j - \alpha(y_j - y_i)$ and $x_i = y_i + \alpha(y_j - y_i)$ are above S_{u_j} and S_{u_i}, respectively, and
ii) there are

$$\tilde{x}_j \in s_{v_j}(x_j) \text{ with } \tilde{x}_j^l = x_j^l,$$
$$\tilde{x}_i \in s_{u_i}(x_i) \text{ with } \tilde{x}_i^l = x_i^l, \text{ and}$$
$$\tilde{y}_i \in s_{v_i}(y_i) \text{ with } \tilde{y}_i^l = y_i^l,$$

satisfying $\tilde{x}_j > \tilde{x}_i > \tilde{y}_i$.

The transfer relating y_i, x_i, y_j and x_j is balanced and the four points are aligned.[37]

[37]Our transfer axiom is not restricted to bundles that are proportional to one another, we choose to highlight that Lemma 2 does not rely on this.

Given that $\tilde{x}_j > \tilde{x}_i > \tilde{y}_i$, let $R_0 \in \mathcal{R}$ be such that

$$\begin{aligned} U(\tilde{y}_i, R_0) &= \{z_i \in X \mid z_i \geqq \tilde{y}_i\}, \\ U(\tilde{x}_i, R_0) &= \{z_i \in X \mid z_i \geqq \tilde{x}_i\}, \\ U(\tilde{x}_j, R_0) &= \{z_j \in X \mid z_j \geqq \tilde{x}_j\}, \text{ and} \\ U(y_j, R_0) &= \{z_j \in X \mid z_j \geqq y_j\}, \end{aligned}$$

so that $y_i \in I(\tilde{y}_i, R_0)$, $x_i \in I(\tilde{x}_i, R_0)$ and $x_j \in I(\tilde{x}_j, R_0)$. Then, $P_{\text{supp}}(y_i, R_0) \subsetneqq P_{\text{supp}}(y_j, R_0)$, and $P_{\text{supp}}(x_i, R_0) = P_{\text{supp}}(x_j, R_0)$. In addition, $w_S(y_i, R_0) = v_i$, $w_S(x_i, R_0) = u_i$, $w_S(x_j, R_0) = u_j$, $w_S(y_j, R_0) = v_j$.

For all $k \in N \setminus \{i, j\}$, pick any $z_k \in \partial S_{u_k}$, and let $R_k \in \mathcal{R}$ be such that $U(z_k, R_k) = \{z'_k \in X \mid z'_k \geqq z_k\}$. Then, $w_S(z_k, R_k) = u_k = v_k$. Let $\Omega = \sum_{k \in N \setminus \{i,j\}} z_k + y_j + y_i = \sum_{k \in N \setminus \{i,j\}} z_k + x_j + x_i$. Finally, let $E \in \mathcal{E}$ be the economy in which the total resources are Ω, both agents i and j have the preference relation R_0 and any other agent $k \in N$ has the preference relation R_k.

Consider the allocation $\tilde{z}_N \in X^N$ such that $\tilde{z}_i = x_i$, $\tilde{z}_j = x_j$ and $\tilde{z}_k = z_k$ for all $k \in N \setminus \{i, j\}$, and the allocation $\tilde{z}'_N \in X^N$ such that $\tilde{z}'_i = y_i$, $\tilde{z}'_j = y_j$ and $\tilde{z}'_k = z_k$ for all $k \in N \setminus \{i, j\}$. By construction, both allocations are Pareto-efficient in E. Then, *same-preference efficiency-neutral transfer* implies that $\tilde{z}_N \mathbf{R}(E) \tilde{z}'_N$. Hence, by *exogenous-reference-set welfarism for same efficiency*, $u \succsim^* v$. □

Conclusion. Let $\mathbf{R}$ be a SOF satisfying the assumptions of Theorem 1. By *continuity* and *exogenous-reference-set welfarism for same efficiency*, the ordering $\succsim^*$ is continuous and monotonic on $\mathbb{R}^N_+$. It is known that an ordering on $\mathbb{R}^N_+$ with such properties additionally satisfies *Hammond equity* if and only if it is the maximin ordering:[38] formally, by Lemma 2, for all $w, v \in \mathbb{R}^N_+$,

$$\begin{aligned} & w \succsim^* v \\ \iff & \min_{i \in N} w_i \geq \min_{i \in N} v_i. \end{aligned}$$

[38] The necessity part of this equivalence is immediate. The sufficiency part can legitimately be referred to as a "folk theorem". Proofs of this result, as a corollary of more general results, can be found in Miyagishima (2017), and Miyagishima et al. (2014) —the latter is a short *corrigendum* concerning the redundancy of the axiom of invariance to permutations adopted in Bosmans and Ooghe (2013). The proof in Bosmans and Ooghe (2013)–Miyagishima et al. (2014), written for a quasi-ordering defined on $\mathbb{R}^N$, applies immediately to a quasi-ordering defined on $\mathbb{R}^N_+$, simply changing "$\frac{1}{k}$" by "$\epsilon_k > 0$ small enough so that $u_{(1)} - \epsilon_k \geq 0$" in the second step of the original proof in Bosmans and Ooghe (2013) (p.155). We note that they actually prove that the equivalence holds even if $\succsim^*$ is assumed to satisfy a weaker property than *Hammond equity*. This result is useful for the proof of Theorem 2.

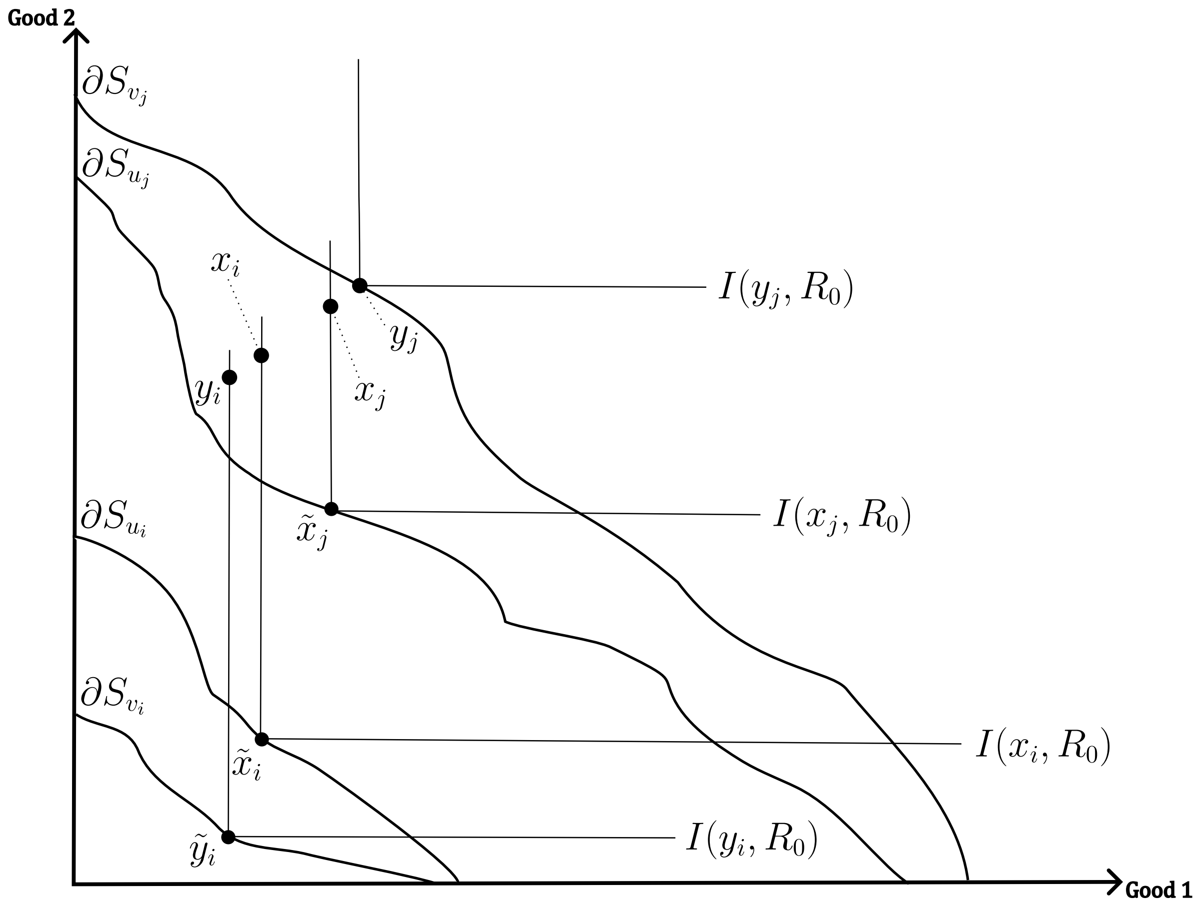


Figure 5. Proof of Lemma 2.

Consider the mappings introduced in the proof of Lemma 1. It is without loss of generality to assume that, for all $w \in \mathbb{R}^N_+$, $W(w) = \min_{i \in N} w_i$. Then, for all $x_N \in X^N$, $\tilde{f}(x_N) = g^{\lambda(x_N)}(\min_{i\in N} w_S(x_i, R_i))$. As a consequence, the mapping

$$\begin{aligned} F^E : D^E_S &\to \mathbb{R} \\ (\lambda, t) &\mapsto g^{\lambda}(t) \end{aligned}$$

satisfies (4); it is continuous, increasing it its second argument, by construction, and increasing in its first argument, by *strong Pareto*. $\square$

7.1.1 Independence of the axioms

We note that all the following SOFs satisfy *anonymity* and *unchanged-contour independence*, which are not assumed in Theorem 1.

Strong Pareto. Take $\mathbf{R}$ a SOF defined on $\mathcal{E}$ for which there is $S \in \mathcal{S}$ such that, for all $E = (R_N, \Omega) \in \mathcal{E}$, for all $x_N, y_N \in X^N$, (7) (just below) is satisfied. **Continuity.**

Take $\mathbf{R}$ a SOF as defined in Section 4.2.2 satisfying (5), involving the leximin ordering. **Same-preference efficiency-neutral transfer.** The SOFs characterized by Bosmans et al. (2018), formally defined in Section 4, satisfy all the axioms of Theorem 1, except this transfer axiom. Let us give another example. Take $\mathbf{R}$ a SOF defined on $\mathcal{E}$ for which there are $S \in \mathcal{S}$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, continuous and increasing, such that, for all $\Omega \in \mathbb{R}^L_{++}$, there is $F^\Omega : \mathbb{R}^2_+ \to \mathbb{R}$, continuous and increasing in each argument, such that for all $E = (R_N, \Omega) \in \mathcal{E}$, all $x_N, y_N \in X^N$, $x_N \mathbf{R}(E) y_N \iff F^\Omega\Big(\lambda^E(x_N), \sum_{i\in N} \varphi(w_S(x_i, R_i))\Big) \geq F^\Omega\Big(\lambda^E(y_N), \sum_{i\in N} \varphi(w_S(y_i, R_i))\Big)$. The mapping φ can be further assumed to be strictly concave. **Exogenous-reference-set welfarism for same efficiency.** Apart from this axiom, the SOF $\mathbf{R}_{\text{eff}}$ satisfies all the axioms of Theorem 1.

7.1.2 Proposition 2

Proposition 2 follows immediately from the fact that $\succsim^*$ is the maximin ordering. Assume by contradiction that $\mathbf{R}$, defined on $\mathcal{E}$, satisfies all the axioms in Proposition 2. By *exogenous-reference set welfarism*, and the above result, for all $E \in \mathcal{E}$, all $x_N, y_N \in X^N$,

$$\begin{aligned} & x_N \mathbf{R}(E) y_N \\ \iff & \min_{i\in N} w_S(x_i, R_i) \geq \min_{i\in N} w_S(y_i, R_i). \end{aligned} \tag{7}$$

This contradicts *strong Pareto.*

7.1.3 Proof of Theorem 2

The proof is a simple modification of the proof of Theorem 1.

The following property is introduced in Bosmans and Ooghe (2013), in an abstract social choice setting. We state it for an ordering $\succsim$ on $\mathbb{R}^N_+$.

Weak Hammond equity

Let $u, v \in \mathbb{R}^N_+$ such that, for some $i \in N$, $v_i = v_{\uparrow_1} < u_i = u_{\uparrow_1} < u_{\uparrow_2} = u_{\uparrow_3} = \cdots = u_{\uparrow_{|N|}} < v_{\uparrow_2} = v_{\uparrow_3} = \cdots = v_{\uparrow_{|N|}}$, then $u \succsim v$.

From Bosmans and Ooghe (2013) and the *corrigendum* in Miyagishima et al. (2014) (see our Footnote 38), we know that an ordering $\succsim$ on $\mathbb{R}^N_+$ is continuous, monotonic and satisfies *weak Hammond equity* if and only if for all $u, v \in \mathbb{R}^N_+$, $u \succsim v \iff \min_{i\in N} u_i \geq \min_{i\in N} v_i$.

Let $\mathbf{R}$ be a SOF defined on $\mathcal{E}$ satisfying same-preference efficiency-neutral uniform inequality reduction and exogenous-reference-set welfarism for same efficiency, with respect to the family of reference sets $S \in \mathcal{S}$.

That the associated ordering $\succsim^*$ on $\mathbb{R}^N_+$ satisfies *weak Hammond equity* follows from an adaptation of the proof of Lemma 2. The following construction is very similar to the one in the proof of Lemma 2, and we include it only for completeness.

Let $u, v \in \mathbb{R}^N_+$ satisfy the assumption of *weak Hammond equity*. For all $k, k' \in N \setminus \{i\}$, let $y = y_k = y'_k \in \partial S_{v_{\uparrow 2}}$ such that $s_{v_i}(y) \neq \emptyset$. Without loss of generality, we assume that $y > \mathbf{0}$.

Let $l \in L$. The choice of y implies that there are $\mathbf{0} \leqq y_i < y$ (close enough to y), above S_{v_i}, and $0 < \alpha < \frac{1}{2(|N|-1)}$ (close enough to $\frac{1}{2(|N|-1)}$), such that

i) $x = y - \alpha(y - y_i)$ and $x_i = y_i + \alpha(y - y_i)(|N|-1)$ are above $S_{u_{\uparrow 2}}$ and S_{u_i}, respectively, and
ii) there are

$$\begin{aligned}
&\tilde{x} \in s_{v_j}(x) \text{ with } \tilde{x}^l = x^l, \\
&\tilde{x}_i \in s_{u_i}(x_i) \text{ with } \tilde{x}_i^l = x_i^l, \text{ and} \\
&\tilde{y}_i \in s_{v_i}(y_i) \text{ with } \tilde{y}_i^l = y_i^l,
\end{aligned}$$

satisfying $\tilde{x} > \tilde{x}_i > \tilde{y}_i$.

Now, let $R_0 \in \mathcal{R}$ be such that $U(\tilde{y}_i, R_0) = \{z_i \in X \mid z_i \geqq \tilde{y}_i\}$, $U(\tilde{x}_i, R_0) = \{z_i \in X \mid z_i \geqq \tilde{x}_i\}$, $U(\tilde{x}, R_0) = \{z \in X \mid z \geqq \tilde{x}\}$, and $U(y, R_0) = \{z \in X \mid z \geqq y\}$. Let $\Omega = (|N|-1)y + y_i = (|N|-1)x + x_i$. Finally, let $E \in \mathcal{E}$ be the economy in which the total resources are given by Ω, and all agents have the preference relation R_0. Consider the allocation $z_N \in X^N$ such that $z_i = x_i$ and $z_k = x$ for all $k \in N \setminus \{i\}$, and the allocation $z'_N \in X^N$ such that $z'_i = y_i$ and $z'_k = y$ for all $k \in N \setminus \{i\}$. By construction, both allocations are Pareto-efficient in E. Then, *same-preference efficiency-neutral uniform inequality reduction* implies that $z_N \mathbf{R}(E) z'_N$. Hence, by *exogenous-reference-set welfarism for same efficiency*, $u \succsim^* v$.

Now, as $\succsim^*$ is the maximin ordering, concluding as in the proof of Theorem 1 ends the proof of Theorem 2. □

7.2 Proof of Theorem 3

The *sufficiency part* is readily checked, and we only prove the *necessity part*. We detail the proof for a SOF satisfying *same-preference efficiency-neutral transfer*. Given the argument in Section 7.1.3, adapting the following proof to the case of *same-preference efficiency-neutral uniform inequality reduction* is easy.

Preliminary step. First, we note that if $\mathbf{S}$ satisfies *lower bound on lower-comprehensiveness around proportional bundles*, then $\mathbf{S}$ satisfies *property A* below. Directly using *Property A* simplifies the exposition of the proof of Lemma 3 below.

Let $\Omega \in \mathbb{R}^L_{++}$. For all bundles $z_i, z_i' \in X$ such that $z_i' > z_i$, we let $\partial S^\Omega(z_i'|z_i) = \partial S^\Omega(z_i') \cap \{y_i \in X \mid y_i > z_i\}$ be the (open) set of bundles of $\partial S^\Omega(z_i')$ that strictly dominate z_i. For all $z_i = \gamma\Omega$ and $z_i' = \gamma'\Omega$, with $0 \leq \gamma < \gamma'$, we let

$$M(z_i', z_i) = \{y_i = \lambda\Omega \in X \mid \text{ there are } l \in L \text{ and } \tilde{z}_i \in \partial S^\Omega(z_i'|z_i) \text{ such that } y_i^l = \tilde{z}_i^l\}.$$

In words, z_i' is a proportional bundle higher than another proportional bundle z_i, and $M(z_i', z_i)$ is the set of all proportional bundles that share a coordinate with a bundle of $\partial S^\Omega(z_i')$ that strictly dominates z_i.

Property A. For all $\Omega \in \mathbb{R}^L_{++}$, for all $z_i = \gamma\Omega > \mathbf{0}$ and all $\alpha > 1$, there exists $\theta > 0$ such that, for all $z_i' = \beta\Omega \geqq \alpha\gamma\Omega$, there is $y_i = \lambda\Omega \in M(z_i', z_i)$ such that

$$\lambda - \beta \geq \theta.$$

In addition (for the case $z_i = \mathbf{0}$), for all $\alpha > 0$, there exists $\theta > 0$ such that, for all $z_i' = \alpha\Omega$, there is $y_i = \lambda\Omega \in M(z_i', \mathbf{0})$ such that $\lambda - \alpha \geq \theta$.[39]

The implication is easy to check and the proof is left to the reader.

Lemma 3. *Let* $\mathbf{R}$ *be a SOF defined on* $\mathcal{E}$ *satisfying same-preference efficiency-neutral transfer and* Ω*-reference-set welfarism for same efficiency, with respect to the system of families of reference sets* $\mathbf{S}$ *that has the lower bound on lower-comprehensiveness property. Then, the associated ordering* $\succsim^*$ *satisfies Hammond equity: let* $u, v \in \mathbb{R}^N_+$ *be such that, for some* $i, j \in N$,

$$v_i < u_i < u_j < v_j, \textit{ and,}$$
$$u_k = v_k \textit{ for all } k \in N \setminus \{i, j\},$$

then, $u \succsim^* v$.

Proof. Let $\Omega \in \mathbb{R}^N_{++}$. Let $z_i = \text{ray}[\mathbf{0}, \Omega) \cap \partial S^\Omega_{u_i} = \beta_i\Omega$ and let $z_j = \text{ray}[\mathbf{0}, \Omega) \cap \partial S^\Omega_{u_j} = \beta_j\Omega$. Let $x_i = \text{ray}[\mathbf{0}, \Omega) \cap \partial S^\Omega_{v_i} = \gamma_i\Omega$ and let $x_j = \text{ray}[\mathbf{0}, \Omega) \cap \partial S^\Omega_{v_j} = \gamma_j\Omega$. Figure 6 illustrates the following construction in the case in which $|L| = 2$.[40] By definition, $\gamma_j > \beta_j > \beta_i > \gamma_i \geq 0$.

[39] Note that given Ω, the lower bound θ depends on the initial parameters γ and α. This property states that for all initial proportional bundle z_i, and all "proportional jump" from this bundle αz_i, there is a minimal gap θ along ray$[\mathbf{0}, \Omega)$, such that, for all other bundle obtained by a "proportional jump" of at least α from z_i, *i.e.* z_i', there is a third proportional bundle, *i.e.* y_i, that i) is above z_i' by at least $\theta\Omega$, and ii) shares at least one coordinate with a bundle on the reference-set-boundary to which z_i' belongs that strictly dominates z_i, *i.e.* $y_i \in M(z_i', z_i)$.

[40] We only represent the bundles and indifference curves of agents i and j.

Property A implies that there is $\theta > 0$ and $y_j = m_j^1 \Omega$ such that $m_j^1 - \beta_j \geq \theta$ and, for some $l \in L$ and some $\tilde{z}_j \in \partial S^{\Omega}(z_j|z_i)$, $y_j^l = \tilde{z}_j^l$. Let $\epsilon^1 > 0$ such that $\beta_i - \epsilon^1 > \gamma_i$, and let $z_j^{\epsilon^1} = (m_j^1 + \epsilon^1)\Omega$, $z_i^{\epsilon^1} = (\beta_i - \epsilon^1)\Omega$. Now, consider $R_0^1 \in \mathcal{R}$ such that

$$\begin{aligned}
&U(\tilde{z}_j, R_0^1) = \{z_j' \in X \mid z_i' \geqq \tilde{z}_j\}, \text{ implying that } y_j \in I(\tilde{z}_j, R_0^1),\\
&U(z_i, R_0^1) = \{z_i' \in X \mid z_i' \geqq z_i\},\\
&U(z_j^{\epsilon^1}, R_0^1) = \{z_j' \in X \mid z_j' \geqq z_j^{\epsilon^1}\}, \text{ and}\\
&U(z_i^{\epsilon^1}, R_0^1) = \{z_i' \in X \mid z_i' \geqq z_i^{\epsilon^1}\}.
\end{aligned}$$

Such a preference relation exists because $\tilde{z}_j \in \partial S^{\Omega}(z_j|z_i)$. In addition, for all $k \in N \backslash \{i, j\}$, let $z_k \in \text{ray}[\mathbf{0}, \Omega) \cap \partial S_{u_k}^{\Omega}$, and let $R_k \in \mathcal{R}$ be such that $U(z_k, R_k) = \{z_k' \in X \mid z_k' \geqq z_k\}$. Let $E^1 \in \mathcal{E}$ be the economy in which agents i and j have the preference relation R_0^1, any other agent k has the preference relation R_k, and total resources are given by Ω. Let

$$w_j^1 = w_{S^{\Omega}}(z_j^{\epsilon^1}, R_0^1) > u_j = w_{S^{\Omega}}(y_j, R_0^1) \text{ and } w_i^1 = w_{S^{\Omega}}(z_i^{\epsilon^1}, R_0^1) < u_i = w_{S^{\Omega}}(z_i, R_0^1),$$

and let $w^1 = (w_j^1, w_i^1, (u_k)_{k \in N \backslash \{i,j\}})$. Let $a_N^1 = (y_j, (z_k)_{k \in N \backslash \{j\}})$ and $z_N^{\epsilon^1} = (z_j^{\epsilon^1}, z_i^{\epsilon^1}, (z_k)_{k \in N \backslash \{i,j\}})$.

In words, there is $\lambda > 0$ such that a_N^1 and $z_N^{\epsilon^1}$ are Pareto-efficient for the economy in which preferences are the same as in E^1 and resources are given by $\lambda\Omega$. Indeed, by construction, there is $\lambda > 0$ such that for all $l \in L$, $y_j^l + \sum_{k \in N \backslash \{j\}} z_k^l = (z_j^{\epsilon^1})^l + (z_i^{\epsilon^1})^l + \sum_{k \in N \backslash \{i,j\}} z_k = \lambda \Omega^l$. In addition, for all $k \in N \setminus \{i, j\}$, $P_{\text{supp}}(y_j, R_0^1) \supsetneqq P_{\text{supp}}(z_i, R_0^1) = P_{\text{supp}}(z_k, R_k)$, and $P_{\text{supp}}(z_j^{\epsilon^1}, R_0^1) = P_{\text{supp}}(z_i^{\epsilon^1}, R_0^1) = P_{\text{supp}}(z_k, R_k)$. Therefore, $a_N^1 \mathbf{I}_{\text{eff}}(E^1) z_N^{\epsilon^1}$.

By *same-preference efficiency-neutral transfer*, $a_N^1 \mathbf{R}(E^1) z_N^{\epsilon^1}$. Thus, *$\Omega$-reference-set welfarism for same efficiency* implies that $u \succsim^* w^1$.

Now, *property A* also implies that there is $\tilde{y}_j = m_j^2 \Omega$ such that $m_j^2 - (m_j^1 + \epsilon^1) \geq \theta$ and, for some $l \in L$ and some $\bar{z}_j \in \partial S^{\Omega}(z_j^{\epsilon^1}|z_i^{\epsilon^1})$, $\tilde{y}_j^l = \bar{z}_j^l$. We refer the reader to Figure 7. Let $\epsilon^2 > 0$ such that $\beta_i - \epsilon^1 - \epsilon^2 > \gamma_i$, and let $z_j^{\epsilon^2} = (m_j^2 + \epsilon^2)\Omega$, $z_i^{\epsilon^2} = (\beta_i - \epsilon^1 - \epsilon^2)\Omega$. Consider $R_0^2 \in \mathcal{R}$ such that

$$\begin{aligned}
&U(\bar{z}_j, R_0^2) = \{z_i' \in X \mid z_i' \geqq \bar{z}_j\},\\
&U(z_i^{\epsilon^1}, R_0^2) = \{z_i' \in X \mid z_i' \geqq z_i^{\epsilon^1}\},\\
&U(z_j^{\epsilon^2}, R_0^2) = \{z_j' \in X \mid z_j' \geqq z_j^{\epsilon^2}\}, \text{ and}\\
&U(z_i^{\epsilon^2}, R_0^2) = \{z_j \in X \mid z_i' \geqq z_i^{\epsilon^1}\}.
\end{aligned}$$

Let $E^2 \in \mathcal{E}$ be the economy in which agents i and j have the preference relation R_0^2, any other agent k has the preference relation R_k, and total resources are given by

Ω. Let $w_i^2 = w_{S^\Omega}(z_i^{\epsilon^2}, R_0^2) > w_j^1 = w_{S^\Omega}(\tilde{y}_j, R_0^1)$, and $w_j^2 = w_{S^\Omega}(z_j^{\epsilon^2}, R_0^2) < w_i^1$, and let $w^2 = (w_j^2, w_i^2, (u_k)_{k\in N\setminus\{i,j\}})$. Let $a_N^2 = (\tilde{y}_j, z_i^{\epsilon^1}, (z_k)_{k\in N\setminus\{i,j\}})$ and $z_N^{\epsilon^2} = (z_j^{\epsilon^2}, z_i^{\epsilon^2}, (z_k)_{k\in N\setminus\{i,j\}})$. Then, $a_N^2 \mathbf{I}_{\text{eff}}(E^2) z_N^{\epsilon^2}$.

By *same-preference efficiency-neutral transfer* $a_N^2 \mathbf{R}(E^2) z_N^{\epsilon^2}$. Thus, *$\Omega$-reference-set welfarism for same efficiency* implies that $w^1 \succsim^* w^2$.

Let $n = \left\lceil \frac{\gamma_j - \beta_j}{\theta} \right\rceil + 1$.[41] By iterating the above construction (at most) n times,[42] and using the transitivity of $\succsim^*$, we obtain that there exists $w \in \mathbb{R}_+^N$ such that

$$
\begin{aligned}
& w = (w_j, w_i, (u_k)_{k\in N\setminus\{i,j\}}) \ \Big(= (w_j, w_i, (v_k)_{k\in N\setminus\{i,j\}})\Big), \\
& w_j > v_j \ , \ w_i > v_i, \text{ and} \\
& u \succsim^* w.
\end{aligned}
$$

By the monotonicity of $\succsim^*$, $w \succsim^* v$, and, by transitivity, $u \succsim^* v$. In the example of Figures 6 and 7, two iterations are sufficient.

□

Conclusion. The analogue of Lemma 1 holds (the proof being the same, after replacing S by S^Ω, for any $\Omega \in \mathbb{R}_{++}^L$):

Let $\mathbf{R}$ be a SOF defined on $\mathcal{E}$ satisfying continuity and Ω-reference-set welfarism for same efficiency, with respect to the system of families of reference sets $\mathbf{S}$. Let $E = (R_N, \Omega) \in \mathcal{E}$. There exists a continuous mapping $f : \bar{D}_{S^\Omega}^E \to \mathbb{R}$ such that, for all $x_N, y_N \in X^N$, $x_N \mathbf{R}(E) y_N \iff f\Big(\lambda^E(x_N), w_{S^\Omega}^{R_N}(x_N)\Big) \geq f\Big(\lambda^E(y_N), w_{S^\Omega}^{R_N}(y_N)\Big)$.

From Lemma 3, we conclude as in the Conclusion step of the proof of Theorem 1. □

Independence of the axioms. Independence is checked as in Section 7.1.1 for all the axioms, taking, when relevant, a system $\mathbf{S}$ that satisfies the lower bound on lower-comprehensiveness property.

7.3 Proof of Theorem 4

The *sufficiency part* is readily checked, and we only prove the *necessity part*. We use some piece of notation introduced in the proof of Theorem 1.

Lemma 4. *Let $\mathbf{R}$ be a SOF defined on $\mathcal{E}$ satisfying independence from proportional expansions, equal-division efficiency-neutral transfer and Ω-reference-set welfarism for same*

[41] For all $t \in \mathbb{R}_+$, $\lceil t \rceil$ denotes the ceiling integer of t.

[42] Let a_j^n, $z_j^{\epsilon^n}$, a_i^n and $z_i^{\epsilon^n}$ denote the bundles allocated to j and i in the two allocations obtained in the nth iteration of the construction described above, respectively. Then, $z_j^{\epsilon^n} > a_j^n > x_j$ and $a_i^n > z_i^{\epsilon^n} > x_i$.

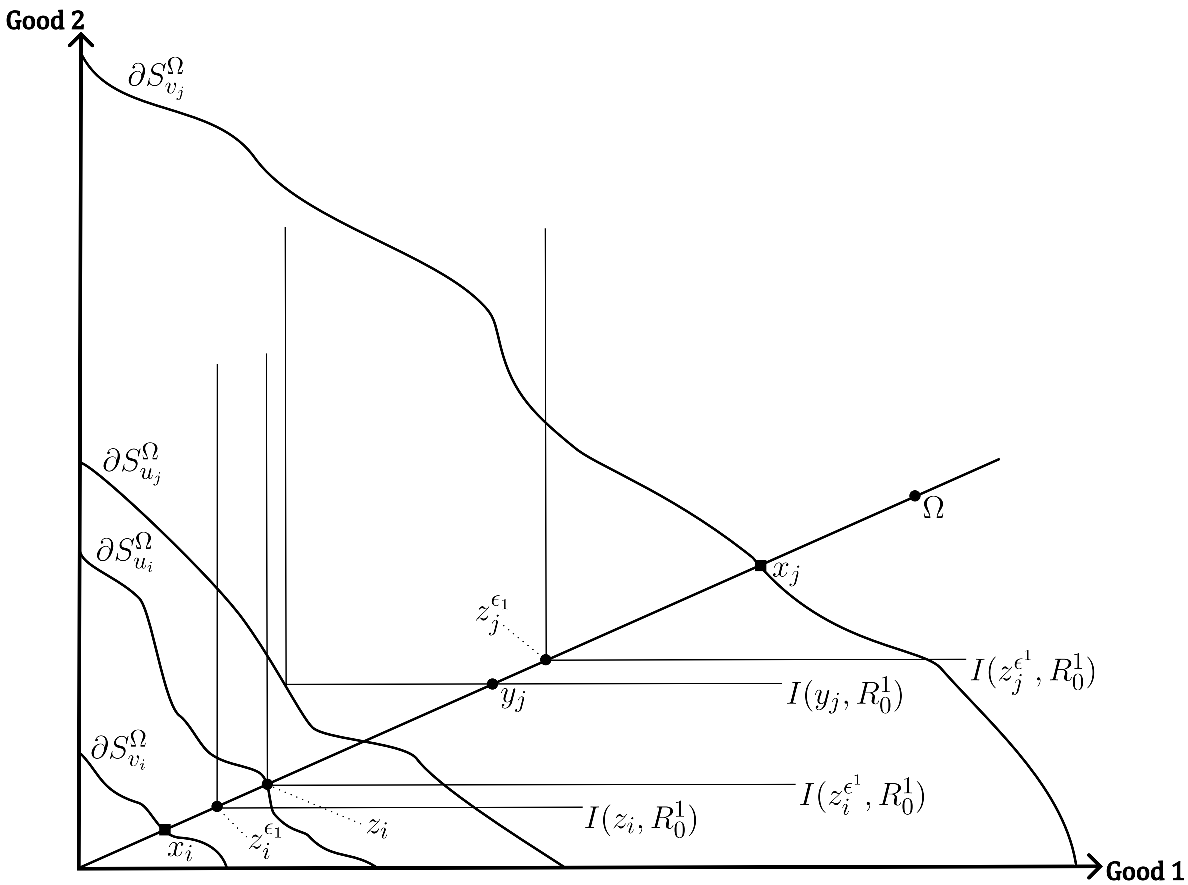


Figure 6. Proof of Lemma 3, first transfer.

efficiency, with respect to the system of families of reference sets $\mathbf{S}$*. Then,* $\mathbf{S}$ *is the* Ω*-ray system.*

Proof. Assume, by contradiction, that there are $\Omega \in \mathbb{R}^L_{++}$ and $\gamma > 0$ such that $S^{\Omega}_{\gamma} \neq \{z \in X | \ z \leqq \gamma\Omega\}$.[43] Let $x = \partial S^{\Omega}_{\gamma} \cap \mathrm{ray}[\mathbf{0}, \Omega)$. Since S^{Ω}_{γ} is lower-comprehensive, this means that there are $l \in L$ and $z \in \partial S^{\Omega}_{\gamma}$ such that $z^l > x^l$. Because the family S^{Ω} is such that $\bigcup_{\mu \in \mathbb{R}_+} \partial S^{\Omega}_{\mu} = X$, the fact that $z^l > x^l$ implies that there are $\gamma' > 0$ (close enough to γ), $z' \in X$, $\epsilon > 0$ and $x^{\epsilon} \in X$ such that

$$
\begin{aligned}
&\gamma' < \gamma \text{ and } S^{\Omega}_{\gamma'} = S^{\Omega}(z'),\\
&|s_{\gamma'}(x)| > 1,\\
&x^l < z'^l < z^l, \text{ and}\\
&x^{\epsilon} = x - \epsilon\Omega > \partial S^{\Omega}_{\gamma'} \cap \mathrm{ray}[\mathbf{0}, \Omega).
\end{aligned}
$$

[43]To avoid confusion in this proof, we index a bundle only when we want to indicate that the bundle is assigned to a specific agent.

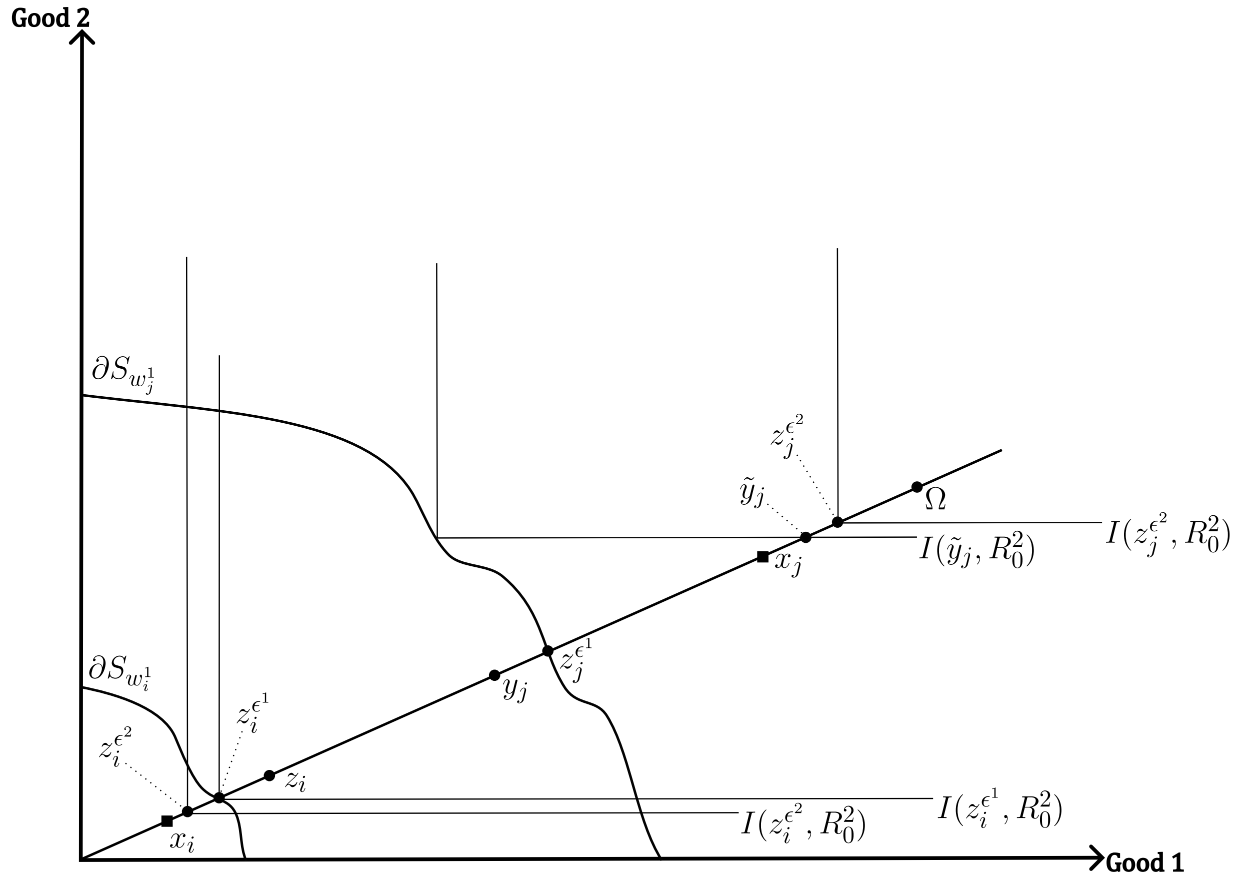


Figure 7. Proof of Lemma 3, second transfer.

Let $y' \in X$ be the bundle on ray$[\mathbf{0}, \Omega)$ such that $y'^l = z'^l$. Note that $y' > x$. Finally, let $\epsilon' > 0$ small enough so that i) $x^{\epsilon'} = x^\epsilon + \epsilon'\Omega < x$ and $y'^{\epsilon'} = y' - \epsilon'\Omega > x$, and ii) there is $z'^{\epsilon'} < z'$, of which the l-coordinate is equal to the l-coordinate of $y'^{\epsilon'}$, such that $s_{\gamma_{\epsilon'}}(x) \neq \emptyset$, where $\gamma_{\epsilon'}$ is defined by $S^\Omega(z'^{\epsilon'}) = S^\Omega_{\gamma_{\epsilon'}}$. The possibility to satisfy ii) follows from the fact that γ' is close enough to γ so that $|s_{\gamma'}(x)| > 1$. We use condition i) to construct a first economy.

Let $i \neq j \in N$ and let $R_i, R_j \in \mathcal{R}$ be such that

$$\begin{aligned}
&U(x^\epsilon, R_j) = \{y \in X \mid y \geqq x^\epsilon\},\\
&U(x^{\epsilon'}, R_j) = \{y \in X \mid y \geqq x^{\epsilon'}\},\\
&U(z', R_i) = \{y \in X \mid y \geqq z'\}, \text{ implying that } y' \in I(z', R_i), \text{ and}\\
&U(z'^{\epsilon'}, R_i) = \{y \in X \mid y \geqq z'^{\epsilon'}\} \text{ implying that } y'^{\epsilon'} \in I(z', R_i).
\end{aligned}$$

Let $\lambda > 0$ such that $x = \frac{\lambda\Omega}{|N|}$. To alleviate notation, let Ω' denote $\lambda\Omega$. For all $k \in N \backslash \{i, j\}$, let $z_k = z_0$, for $z_0 \in$ ray$[\mathbf{0}, \Omega)$ such that $z_0 > x$, and let $R_k = R_0 \in \mathcal{R}$, for R_0 such that $U(z_0, R_0) = \{y \in X \mid y \geqq z_0\}$. Let $a_N \in X$ be defined by $a_j = x^\epsilon$, $a_i = y'$ and, for all $k \in N \setminus \{i,j\}$, $a_k = z_0$, and let $b_N \in X$ be defined by $b_j = x^{\epsilon'}$, $b_i = y'^{\epsilon'}$ and, for all $k \in N \setminus \{i,j\}$, $b_k = z_0$.

Let $E \in \mathcal{E}$ be the economy in which i has the preference R_i, j has the preference R_j, and all other agents have the preference R_0, and resources are given by Ω. Let $E' \in \mathcal{E}$ be the economy in which preferences are the same as in E, and resources are given by Ω'. By *independence from proportional expansions*, $a_N \mathbf{R}(E) b_N \iff a_N \mathbf{R}(E') b_N$. Moreover, $a_N \mathbf{I}_{\text{eff}}(E') b_N$. Then, *equal-division efficiency-neutral transfer* implies that $b_N \mathbf{P}(E) a_N$.

Since $\mathbf{R}_{\text{eff}}$ also satisfies *independence from proportional expansions*, $a_N \mathbf{I}_{\text{eff}}(E) b_N$. Letting $u \in \mathbb{R}^N_+$ be defined by $u_i = w_{S^\Omega}(a_i, R_i)$ $(= \gamma')$, $u_j = w_{S^\Omega}(a_j, R_j)$ and, for all $k \in N$, $u_k = w_{S^\Omega}(a_k, R_0)$, and letting $v \in \mathbb{R}^N_+$ be defined by $v_i = w_{S^\Omega}(b_i, R_i)$ $(= \gamma_{\epsilon'})$, $v_j = w_{S^\Omega}(b_j, R_j)$ and, for all $k \in N$, $v_k = w_{S^\Omega}(b_k, R_0)$, the axiom of *$\Omega$-reference-set welfarism for same efficiency* yields $v \succ^* u$.

Note that, for all $k \in N \setminus \{i,j\}$, $v_k = u_k$, and $v_j > u_j > u_i > v_i$. In addition, $x^{\epsilon'} > \mathbf{0}$, $x^{\epsilon'} \in \partial S^\Omega_{v_j}$ and, by the property ii) of $\gamma_{\epsilon'}$, $s_{v_i}(x^{\epsilon'}) \neq \emptyset$. Thus, we can use the same construction of allocations and preferences as the construction in the proof of Lemma 2, except that the allocations related by the transfer are proportional to Ω. This yields the existence of a preference profile $R_N \in \mathcal{R}$ and two allocations $\tilde{a}_N, \tilde{b}_N \in X^N$ such that, in $\tilde{E} = (R_N, \Omega)$, $\tilde{a}_N \mathbf{I}_{\text{eff}}(\tilde{E}) \tilde{b}_N$ and

$$\begin{aligned} &\tilde{a}_k = \tilde{b}_k \text{ for all } k \in N \setminus \{i,j\}, \\ &\tilde{b}_j - \Delta = \tilde{a}_j > \tilde{a}_i = \tilde{b}_i + \Delta, \text{ for some } \Delta \in \mathbb{R}^L_{++}, \end{aligned}$$

and $w_{S^\Omega}(\tilde{b}_j, R_j) = v_j$, $w_{S^\Omega}(\tilde{a}_j, R_j) = u_j$, $w_{S^\Omega}(\tilde{a}_i, R_i) = u_i$, $w_{S^\Omega}(\tilde{b}_i, R_i) = v_i$. Let $\bar{\lambda} > 0$ such that $\tilde{a}_j > \frac{\bar{\lambda}\Omega}{|N|} > \tilde{a}_i$, and let $\bar{E} = (R_N, \bar{\lambda}\Omega)$. By *equal-division efficiency-neutral transfer* and *independence from proportional expansions*, $\tilde{a}_N \mathbf{R}(\tilde{E}) \tilde{b}_N$. But then, *$\Omega$-reference-set welfarism for same efficiency* implies that $u \succsim^* v$, which is a contradiction. $\square$

Lemma 5. *Let $\mathbf{R}$ be a SOF defined on $\mathcal{E}$ satisfying independence from proportional expansions, equal-division efficiency-neutral transfer and Ω-reference-set welfarism for same efficiency, with respect to the system of families of reference sets $\mathbf{S}$. Then, the associated ordering $\succsim^*$ satisfies Hammond equity: let $u, v \in \mathbb{R}^N_+$ be such that, for some $i, j \in N$,*

$$\begin{aligned} &v_i < u_i < u_j < v_j, \text{ and,} \\ &u_k = v_k \text{ for all } k \in N \setminus \{i,j\}, \end{aligned}$$

then, $u \succsim^* v$.

Proof. By Lemma 4, $\mathbf{S}$ is the Ω-ray system. Let $\Omega \in \mathbb{R}^N_{++}$. Let $z_i = \text{ray}[\mathbf{0}, \Omega) \cap \partial S^\Omega_{u_i} = u_i \Omega$

and let $z_j = \text{ray}[\mathbf{0}, \Omega) \cap \partial S^{\Omega}_{u_j} = u_j\Omega$. Let $x_i = \text{ray}[\mathbf{0}, \Omega) \cap \partial S^{\Omega}_{v_i} = v_i\Omega$ and let $x_j = \text{ray}[\mathbf{0}, \Omega) \cap \partial S^{\Omega}_{v_j} = v_j\Omega$.

Let $l \in L$, and let $\theta > 0$ be such that $z_i^l - \theta > 0$. Note that the bundle defined by replacing z_i^l by $z_i^l - \theta$ also belongs to $\partial S^{\Omega}_{u_i}$. Let $0 < \epsilon^1 < \theta$ arbitrarily small, so that, defining $\Delta^1 \in \mathbb{R}^L_{++}$ by $(\Delta^1)^l = \theta$, $(\Delta^1)^g = \epsilon^1\Omega^g$, for all $g \in L \setminus \{l\}$, one has $z_{i,1} = z_i - \Delta^1 > x_i$. Let $z_{j,1} = z_j + \Delta^1$. Let $R_0 \in \mathcal{R}$ be such that

$$\begin{aligned}
&U(z_i, R_0) = \{y_i \in X \mid y_i \geqq z_i\},\\
&U(z_{i,1}, R_0) = \{y_i \in X \mid y_i \geqq z_{i,1}\},\\
&U(z_j, R_0) = \{y_j \in X \mid y_j \geqq z_j\}, \text{ and}\\
&U(z_{j,1}, R_0) = \{y_j \in X \mid y_j \geqq z_{j,1}\}.
\end{aligned}$$

In addition, for all $k \in N \setminus \{i, j\}$, let $z_k \in \text{ray}[\mathbf{0}, \Omega) \cap \partial S^{\Omega}_{u_k}$, and let $R_k \in \mathcal{R}$ be such that $U(z_k, R_k) = \{z'_k \in X \mid z'_k \geqq z_k\}$. Let $\mu_1 > 0$ such that $z_i < \frac{\mu_1\Omega}{|N|} < z_j$, and let $E^1 \in \mathcal{E}$ be the economy in which agents i and j have the preference relation R_0 and all the other agents k have the preference relation R_k, and total resources are given by $\mu_1\Omega$. Let $a_{N,1} \in X$ be defined by $a_{i,1} = z_i$, $a_{j,1} = z_j$ and, for all $k \in N \setminus \{i, j\}$, $a_{k,1} = z_k$, and let $b_{N,1} \in X$ be defined by $b_{i,1} = z_{i,1}$, $b_{j,1} = z_{j,1}$ and, for all $k \in N \setminus \{i, j\}$, $b_{k,1} = z_k$. Then, $a_{N,1}\mathbf{I}_{\text{eff}}(E^1)b_{N,1}$.

Let $\tilde{z}_{j,1} = \lambda_{j,1}\Omega$ and $\tilde{z}_{i,1} = \lambda_{i,1}\Omega$ be the proportional bundles such that $\tilde{z}^l_{j,1} = z^l_{j,1}$ and $\tilde{z}^g_{i,1} = z^g_{i,1} - \epsilon^1\Omega^g$, for all $g \in L \setminus \{l\}$. Then, $\lambda_{j,1} - u_j = \theta$ and $u_i - \lambda_{i,1} = \epsilon^1$.

For all $k \in N \setminus \{i, j\}$, $w_{S^\Omega}(a_{k,1}, R_k) = w_{S^\Omega}(b_{k,1}, R_k) = u_k = v_k$, $w_{S^\Omega}(a_{i,1}, R_0) = u_i$, $w_{S^\Omega}(b_{i,1}, R_0) = u_i - \epsilon^1$, $w_{S^\Omega}(a_{j,1}, R_0) = u_j$, $w_{S^\Omega}(b_{j,1}, R_0) = u_j + \theta$. Let $w^1 = (u_i - \epsilon^1, u_j + \theta, (u_k)_{k\in N\setminus\{i,j\}})$.

Therefore, by *independence from proportional expansions*, *equal-division efficiency-neutral transfer* and *Ω-reference-set welfarism for same efficiency*, $u \succsim^* w^1$.

Now, let $0 < \epsilon^2 < \theta$ arbitrarily small, so that, defining $\Delta^2 \in \mathbb{R}^L_{++}$ by $(\Delta^2)^l = \theta$, $(\Delta^2)^g = \epsilon^2\Omega^g$, for all $g \in L \setminus \{l\}$, one has $z_{i,2} = \tilde{z}_{i,1} - \Delta^2 > x_i$. Let $z_{j,2} = \tilde{z}_{j,1} + \Delta^2$.

Let $w^2 = (w^1_i - \epsilon^2, w^1_j + \theta, (u_k)_{k\in N\setminus\{i,j\}})$. Then, $w^1 \succsim^* w^2$. We omitted the details, which follow exactly the same logic as for the case of E^1 just above.

Let $n = \left\lceil \frac{v_j - u_j}{\theta} \right\rceil + 1$. By iterating the above construction (at most) n times,[44] and using

[44]Let $a_{N,n}$ and $b_{N,n}$ be the allocations then obtained. It holds that $b_{j,n} > a_{j,n} > x_j$ and $a_{i,n} > b_{i,n} > x_i$.

the transitivity of $\succsim^*$, we obtain that there exists $w \in \mathbb{R}^N_+$ such that

$$w = (w_i, w_j, (u_k)_{k\in N\setminus\{i,j\}}) \ \Big(= (w_i, w_j, (v_k)_{k\in N\setminus\{i,j\}})\Big),$$
$$w_i > v_i \ , \ w_j > v_j, \text{ and}$$
$$u \succsim^* w.$$

By the monotonicity of $\succsim^*$, $w \succsim^* v$, and, by transitivity, $u \succsim^* v$. □

Combining Lemma 4 and Lemma 5, we conclude as in the Conclusion step of the proof of Theorem 3. □

7.3.1 Independence of the axioms

We note that all the following SOFs satisfy *anonymity*. Given what precedes, we omit the cases of *strong Pareto*, *continuity* and *equal-division efficiency-neutral transfer*.

Independence from proportional expansions. Take $\mathbf{R}$ a SOF defined on $\mathcal{E}$ for which there is $0 < \alpha < 1$ such that, for all $E = (R_N, \Omega) \in \mathcal{E}$, all $x_N, y_N \in X^N$, $x_N\mathbf{R}(E)y_N$ if and only if $\frac{\alpha}{||\Omega||}\lambda^E(x_N)+(1-\alpha)\min_{i\in N} w_\Omega(x_i, R_i) \geq \frac{\alpha}{||\Omega||}\lambda^E(x_N)+(1-\alpha)\min_{i\in N} w_\Omega(x_i, R_i)$, where $||\cdot||$ denotes the ℓ^2-norm mapping. **Ω-reference-set welfarism for same efficiency.** Let $\mathcal{R}_H$ denote the subset of strictly convex, strictly monotonic and homothetic preferences (defined in Section 4.2). Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, continuous, increasing and strictly concave. Let $0 < \alpha < 1$. Take $\mathbf{R}$ a SOF defined on $\mathcal{E}$ such that, for all $E = (R_N, \Omega)$, i) if for all $i \in N$, $R_i = R_0$ with $R_0 \in \mathcal{R}_H$, then, for all $x_N, y_N \in X^N$, $x_N\mathbf{R}(E)y_N$ if and only if $\sum_{i\in N}\varphi(w_\Omega(x_i, R_0)) \geq \sum_{i\in N}\varphi(w_\Omega(y_i, R_0))$, and ii) otherwise, for all $x_N, y_N \in X^N$, $x_N\mathbf{R}(E)y_N$ if and only if $\alpha\lambda^E(x_N) + (1-\alpha)\min_{i\in N} w_\Omega(x_i, R_i) \geq \alpha\lambda^E(y_N) + (1-\alpha)\min_{i\in N} w_\Omega(y_i, R_i)$. Note that this SOF violates *Ω-reference-set welfarism for same efficiency*, but also *unchanged-contour independence*.

7.4 Monotonicity of the ordering defined on welfare indices

We assume that $\succsim^*$ is monotonic on $\mathbb{R}^N_+$, which may seem unnecessary given that $\mathbf{R}_{\text{eff}}$ satisfies *strong Pareto*. Let *exogenous-reference-set welfarism for same efficiency (bis)* be defined by removing the monotonicity assumption in *exogenous-reference-set welfarism for same efficiency*. It is true that *same-preference efficiency-neutral transfer* and *exogenous-reference-set welfarism for same efficiency (bis)* are compatible with $\succsim^*$ being an inequality measure. For example, it would be possible for $\succsim^*$ to satisfy, for all $w, v \in \mathbb{R}^N_+ \setminus \{\mathbf{0}\}$, $w \succsim^* v \iff \frac{\min_{i\in N} w_i}{\max_{i\in N} w_i} \geq \frac{\min_{i\in N} v_i}{\max_{i\in N} v_i}$, and $w \succsim^* \mathbf{0}$.

However, monotonicity is a very natural property when we assume that $\mathbf{R}(E)$ is sensitive to the associated welfare indices, even when it compares non-equally efficient allocations, in a way that actually rules out any sensible inequality measure. Let us fromalize this point.

The following property is defined for a SOF satisfying *exogenous-reference-set welfarism for same efficiency (bis)*, with respect to the ordering $\succsim^*$ and the family of reference sets S. It requires that, in the social evaluation, an improvement in terms of reference-set welfare indices cannot be superseded by an *arbitrarily small* improvement of efficiency:

Non-vanishing sensitivity to individual welfare indices

Let $\lambda \in \mathbb{R}_+$ and let $w, v \in \mathbb{R}_+^N$ be such that $w \succ^* v$. There is $\epsilon \in \mathbb{R}_+$ such that for all $E = (R_N, \Omega) \in \mathcal{E}$, $x_N, y_N \in X^N$ satisfying

$$\begin{aligned} &\lambda = \lambda^E(y_N) < \lambda^E(x_N) \leq \lambda^E(y_N) + \epsilon, \text{ and} \\ &w_S^{R_N}(y_N) = w, w_S^{R_N}(x_N) = v, \end{aligned}$$

$y_N \mathbf{R}(E) x_N$.[45]

Note that ϵ above depends on the triplet (λ, w, v). This property states that it does not hold that, for any (arbitrarily small) $\beta > 0$, there are an economy E, and two allocations x_N, y_N such that the level of efficiency of x_N exceeds that of y_N by (only) β, while x_N is considered worse than y_N according to the pair $(\succsim^*, S)$, and $x_N \mathbf{P}(E) y_N$.

Exogenous-reference-set welfarism for same efficiency (bis) rules out the possibility for the SOF $\mathbf{R}$ to be completely insensitive to welfare indices. It is important to note that *non-vanishing sensitivity to individual welfare indices* is compatible with the sensitivity of $\mathbf{R}$ being arbitrarily small, as long as it is not possible to construct a sequence of economies such that this sensitivity vanishes. The following example sheds more light on this point.

Example. Let $0 < \alpha < 1$, and let the SOF $\mathbf{R}_\alpha$ be such that *for all* $E \in \mathcal{E}$, for all $x_N, y_N \in X^N$,

$$\begin{aligned} &x_N \mathbf{R}_\alpha(E) y_N \\ \iff &\frac{\alpha}{|N|}\lambda^E(x_N) + (1-\alpha)\min_{i \in N} w_S(x_i, R_i) \geq \frac{\alpha}{|N|}\lambda^E(y_N) + (1-\alpha)\min_{i \in N} w_S(y_i, R_i). \end{aligned}$$

By *exogenous-reference-set welfarism for same efficiency (bis)*, α must indeed be lower than 1. However, for any other value of α, $\mathbf{R}_\alpha$ satisfies *non-vanishing*

[45] While both *same-preference efficiency-neutral transfer* and *exogenous-reference-set welfarism for same efficiency (bis)* are compatible with λ^E being either an ordinal or a cardinal variable, *non-vanishing sensitivity to reference-set indices* is based on λ^E being cardinal.

sensitivity to individual welfare indices.[46] Now, let $\tilde{\mathbf{R}}_\alpha$ be such that there are a sequence $(E_n) \in \mathcal{E}^{\mathbb{N}}$, and a sequence $(\alpha_n) \in]0,1[^{\mathbb{N}}$, with $\lim_n \alpha_n = 1$, such that, for all $n \in \mathbb{N}$, for all $x_N, y_N \in X^N$,

$$\begin{aligned} & x_N \tilde{\mathbf{R}}_\alpha(E_n) y_N \\ \iff & \frac{\alpha_n}{|N|}\lambda^{E_n}(x_N) + (1-\alpha_n)\min_{i\in N} w_S(x_i, R_{i,n}) \geq \frac{\alpha_n}{|N|}\lambda^{E_n}(y_N) + (1-\alpha_n)\min_{i\in N} w_S(y_i, R_{i,n}). \end{aligned}$$

The SOF $\tilde{\mathbf{R}}_\alpha$ violates *non-vanishing sensitivity to individual welfare indices.*

Non-vanishing sensitivity to individual welfare indices turns out to imply that $\succsim^*$ cannot be any sensible inequality measure:

Proposition 3. *Let* $\mathbf{R}$ *be a SOF satisfying strong Pareto, exogenous-reference-set welfarism for same efficiency (bis), and non-vanishing sensitivity to individual welfare indices, with respect to the family of reference sets* S *and the ordering* $\succsim^*$*. Let* $w \in \mathbb{R}^N_+$ *and* $k \in \arg\max_{i\in N} w_i$*. Then, for all* $\Delta > 0$, $(w_1, \ldots, w_k+\Delta, \ldots, w_{|N|}) \succsim^* (w_1, \ldots, w_k, \ldots, w_{|N|})$.

Because k is one of the best-off individuals according to w, the inequality is unequivocally increased in $(w_1, \ldots, w_k + \Delta, \ldots, w_{|N|})$. Furthermore, note that Δ can be arbitrarily large. Clearly, an inequality measure should not consider $(w_1, \ldots, w_k + \Delta, \ldots, w_{|N|})$ as a (weak) social improvement compared to w.

We conclude our discussion of monotonicity by combining *non-vanishing sensitivity to individual welfare indices* and local non-satiation. In our view, local non-satiation, which is violated by standard inequality measures, becomes a very mild assumption in view of Proposition 3, which rules out any sensible inequality measures. The ordering $\succsim$ on $\mathbb{R}^N_+$ is **locally non-satiated** if, for all $w \in \mathbb{R}^N_+$, for all $\epsilon > 0$, there is $v \in B(w,\epsilon) \cap \mathbb{R}^N_+$ such that $v \succ w$.

Proposition 4. *Let* $\mathbf{R}$ *be a SOF satisfying strong Pareto, exogenous-reference-set welfarism for same efficiency (bis), and non-vanishing sensitivity to individual welfare indices, with respect to the family of reference sets* S *and the locally non-satiated ordering* $\succsim^*$*. Let* $w \in \mathbb{R}^N_+$ *and let* $\Delta \in \mathbb{R}^N_{++}$*. Then,* $w + \Delta \succ^* w$*.*

Propositions 3 and 4 provide a strong justification for the monotonicity assumption in *exogenous-reference-set welfarism for same efficiency.*

[46] Let $(\lambda, w, v) \in \mathbb{R}_+ \times \mathbb{R}^N_+ \times \mathbb{R}^N_+$ satisfy the assumptions of the axiom, with $\min_{i\in N} w_i = \min_{i\in N} v_i + t$, for some $t > 0$. Then, $\epsilon = \frac{|N|(1-\alpha)t}{\alpha} > 0$ is such that the conclusion of the *non-vanishing sensitivity to individual welfare indices* holds.

7.4.1 Proof of Proposition 3

Let $\mathbf{R}$, $w \in \mathbb{R}^N_+$ and $k \in N$ satisfy the assumptions of Proposition 3. Let $\Delta > 0$, and let $w^\Delta = (w_1, \cdots, w_k + \Delta, \cdots, w_{|N|})$. By contradiction, assume that $w \succ^* w^\Delta$. By *non-vanishing sensitivity to individual welfare indices*, there is $\epsilon^* \in \mathbb{R}_+$ such that for all $E = (R_N, \Omega) \in \mathcal{E}$, for all $x_N \in \text{Par}(E)$, and all y_N satisfying

$$
\begin{aligned}
&1 - \epsilon^* \leq \lambda^E(y_N) < 1, \text{ and} \\
&w_S^{R_N}(y_N) = w, w_S^{R_N}(x_N) = w^\Delta,
\end{aligned}
$$

$y_N \mathbf{R}(E) x_N$. The following construction is illustrated in Figure 8 for the case in which $|L|= 2$ and $N = \{i, j, k\}$. We use some piece of the notation introduced in the proof of Theorem 1.

Let $z_k \in \partial S_{w_k+\Delta}$ be such that $s_{w_k}(z_k) \neq \emptyset$. Without loss of generality, we assume that $z_k > \mathbf{0}$. For all $j \in N \setminus \{k\}$, let $z_j = \text{seg}[\mathbf{0}, z_k] \cap \partial S_{w_j}$, and let $R_j \in \mathcal{R}$ be such that $U(z_j, R_j) = \{z'_j \in X \mid z'_j \geqq z_j\}$. Let $\Omega = \sum_{i \in N} z_i$.

The choice of z_k implies that there exist $y_k \in \partial S_{w_k}$ with the subsequent properties. First, $z_k > y_k$. Second, letting $\omega_{y_k} = \text{seg}[\mathbf{0}, z_k] \cap \{y'_k \in X | y'_k \geqq y_k\}$ and letting λ_{y_k} be the real number such that $\omega_{y_k} = \lambda_{y_k} z_k$, one has $\lambda_{y_k} \geq 1 - \epsilon^*$.

Let $R_k \in \mathcal{R}$ be such that $U(z_k, R_k) = \{z'_k \in X \mid z'_k \geqq z_k\}$ and $U(y_k, R_k) = \{y'_k \in X \mid y'_k \geqq y_k\}$. Then, for all $j \in N \setminus \{k\}$, $P_{\text{supp}}(z_j, R_j) = P_{\text{supp}}(z_k, R_k) = P_{\text{supp}}(y_k, R_k)$. Let $R_N = (R_k, (R_j)_{j \in N \setminus \{k\}})$. Then, $z_N \in \text{Par}(E)$ and $1 > \lambda^E((y_k, z_{-k})) \geq 1 - \epsilon^*$. Furthermore, $w_S(z_k, R_k) = w_k + \Delta$, $w_S(y_k, R_k) = w_k$ and, for all $j \in N \setminus \{k\}$, $w_S(z_j, R_j) = w_j$. Recall that, by assumption, $w \succ^* w^\Delta$. By *non-vanishing sensitivity to individual welfare indices*, $(y_k, z_{-k}) \mathbf{R}(E) z_N$. However, by *strong Pareto*, $z_N \mathbf{P}(E)(y_k, z_{-k})$; a contradiction. $\square$

7.4.2 Proof of Proposition 4

Let $\mathbf{R}$ be a SOF satisfying the assumptions of Proposition 4. Let $w > v \in \mathbb{R}^N_+$; we prove that $w \succ^* v$. By local non-satiation, there is $\epsilon > 0$ (small enough) such that there exists $v' \in B(v, \epsilon) \cap \mathbb{R}^N_+$ satisfying $v' \succ^* v$ and $w = v' + (\Delta_i)_{i \in N}$, where $(\Delta_i)_{i \in N} \in \mathbb{R}^N_{++}$. Then, iteratively applying, for each $i \in N$, the argument in the proof of Proposition 3, we conclude that $w \succsim^* v'$, which, by transitivity, yields the result. $\square$

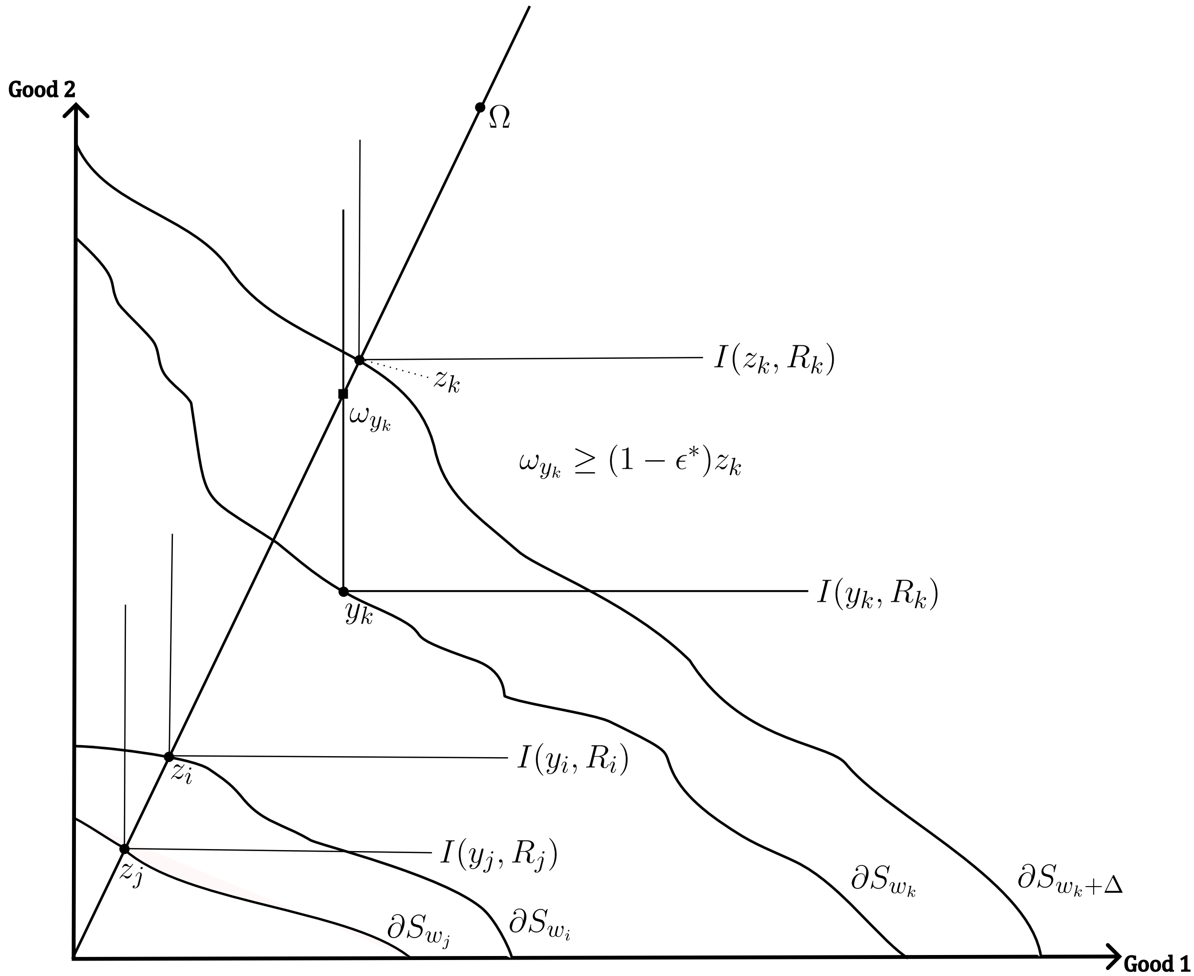


Figure 8. Proof of Proposition 3.

References

Amiel, Y., Cowell, F., 1999. Thinking about inequality: Personal judgment and income distributions. Cambridge University Press.

Bosmans, K., Decancq, K., Ooghe, E., 2018. Who's afraid of aggregating money metrics? Theoretical Economics 13, 467–484.

Bosmans, K., Ooghe, E., 2013. A characterization of maximin. Economic Theory Bulletin 1, 151–156.

Bossert, W., Kamaga, K., 2020. An axiomatization of the mixed utilitarian–maximin social welfare orderings. Economic Theory 69, 451–473.

Brouillette, J.F., Jones, C.I., Klenow, P.J., 2025. Race and economic well-being in the united states. American Economic Review: Insights 7, 429–46.

Capeau, B., Decoster, A., De Sadeleer, L., 2024. Interpersonal comparisons by means of money metric utilities: why one should use the same reference prices for all. Working paper.

Dalton, H., 1920. The measurement of the inequality of incomes. The Economic Journal 30, 348–361.

De Rock, B., Moramarco, D., 2025. Multidimensional fairness with respect for individual preferences. Social Choice and Welfare , 1–27.

Deaton, A., 1979. The distance function in consumer behaviour with applications to index numbers and optimal taxation. The Review of Economic Studies 46, 391–405.

Deaton, A., Muellbauer, J., 1980. Economics and consumer behavior. Cambridge university press.

Debreu, G., 1951. The coefficient of resource utilization. Econometrica: Journal of the Econometric Society , 273–292.

Decancq, K., Nys, A., 2021. Non-parametric well-being comparisons. European Economic Review 133, 103666.

Decancq, K., Schokkaert, E., 2016. Beyond gdp: Using equivalent incomes to measure well-being in europe. Social indicators research 126, 21–55.

Dworkin, R., 1981. What is equality? part 2: Equality of resources. Philosophy & Public Affairs 10, 64–84.

Dworkin, R., 2002. Sovereign virtue: The theory and practice of equality. Harvard university press.

Fleurbaey, M., 2008. Fairness, responsibility, and welfare. Oxford University Press.

Fleurbaey, M., Luchini, S., Muller, C., Schokkaert, E., 2013. Equivalent income and fair evaluation of health care. Health Economics 22, 711–729.

Fleurbaey, M., Maniquet, F., 2011. A theory of fairness and social welfare. volume 48. Cambridge University Press.

Fleurbaey, M., Maniquet, F., 2017a. Addendum to “fairness and well-being measurement”. Mathematical Social Sciences 90, 127–128.

Fleurbaey, M., Maniquet, F., 2017b. Fairness and well-being measurement. Mathematical Social Sciences 90, 119–126.

Fleurbaey, M., Maniquet, F., 2018. Inequality-averse well-being measurement. International Journal of Economic Theory 14, 35–50.

Fleurbaey, M., Tadenuma, K., 2014. Universal social orderings: An integrated theory of policy evaluation, inter-society comparisons, and interpersonal comparisons. Review of Economic Studies 81, 1071–1101.

Fleurbaey, M., Trannoy, A., 2003. The impossibility of a paretian egalitarian. Social Choice and Welfare 21, 243–263.

Foley, D.K., 1967. Resource allocation and the public sector. Yale University.

Hammitt, J.K., 2021. Accounting for the distribution of benefits and costs in benefit–cost analysis. Journal of Benefit-Cost Analysis 12, 64–84.

Hammond, P.J., 1976. Equity, arrow's conditions, and rawls' difference principle. Econometrica: Journal of the Econometric Society , 793–804.

Jones, C.I., Klenow, P.J., 2016. Beyond gdp? welfare across countries and time. American Economic Review 106, 2426–2457.

Kamaga, K., 2018. When do utilitarianism and egalitarianism agree on evaluation? an intersection approach. Mathematical social sciences 94, 41–48.

Kiguchi, N., Nakamura, K., 2026. Utilitarianism or egalitarianism? a new characterization of the mixed utilitarian-maximin social welfare orderings: N. kiguchi, k. nakamura. Social Choice and Welfare 66, 109–126.

King, M.A., 1983. Welfare analysis of tax reforms using household data. Journal of Public Economics 21, 183–214.

Kirman, A.P., 1992. Whom or what does the representative individual represent? Journal of economic perspectives 6, 117–136.

Kolm, S.C., 1969. The optimal production of social justice. Public economics , 145–200.

Layard, P.R.G., Layard, R., Glaister, S., 1994. Cost-benefit analysis. Cambridge University Press.

Maniquet, F., Moramarco, D., 2022. On the measurement of well-being with reference consumption. Working paper.

Maniquet, F., Moramarco, D., 2025. Relative multi-dimensional poverty measurement and deaton's distance function. The Journal of Economic Inequality , 1–12.

Miyagishima, K., 2017. A characterization of the maximin social ordering. Economics Bulletin 30, 1278–1282.

Miyagishima, K., Bosmans, K., Ooghe, E., 2014. A characterization of maximin: corrigendum. Economic Theory Bulletin 2, 219–220.

OECD, 2011–2024. How's Life? Measuring Well-being. OECD Publishing.

Pazner, E.A., Schmeidler, D., 1978. Egalitarian equivalent allocations: A new concept of economic equity. The Quarterly Journal of Economics 92, 671–687.

Piacquadio, P.G., 2017. A fairness justification of utilitarianism. Econometrica 85, 1261–1276.

Pigou, A.C., 1912. The Economics of Welfare. Macmillan, London.

Ravallion, M., 2012. Mashup indices of development. The World Bank Research Observer 27, 1–32.

Rawls, J., 1971. A theory of justice. 2. The Law of Peoples 67.

Roberts, K.W., 1980. Interpersonal comparability and social choice theory. The Review of Economic Studies 47, 421–439.

Steinhaus, H., 1948. The problem of fair division. Econometrica 16, 101–104.

Stiglitz, J.E., Sen, A., Fitoussi, J.P., 2009. Report by the commission on the measurement of economic performance and social progress.

Thomson, W., 1994. Notions of equal, or equivalent, opportunities. Social Choice and Welfare 11, 137–156.

Thomson, W., 2011. Fair allocation rules, in: Handbook of social choice and welfare. Elsevier. volume 2, pp. 393–506.

United Nations, 2024. Pact for the future, global digital compact and declaration on future generations. Adopted at the Summit of the Future, New York.

Varian, H., 1974. Efficiency, equity and envy. Journal of Economic Theory 9, 63–91.

Yang, L., 2018. Measuring well-being: a multidimensional index integrating subjective well-being and preferences. Journal of Human Development and Capabilities 19, 456–476.